CHAPTER 4

THE DYNAMICS OF ELECTRON BEAMS

ROY W. GOULD, Editor

4-1 Introduction. (Sections 4-1 through 4-7 based on remarks by John Whinnery.) The subject of this chapter is a special class of problems in plasma dynamics which has received extensive study by workers in the field of microwave electron tube research. The plasma is in the form of a beam of electrons whose directed velocity is generally much greater than the random velocities of the electrons, since these electrons come from a cathode whose temperature is 1000 °K and they are accelerated through a potential difference of a few hundred to a few thousand volts. Electron beams do not conform to the usual definition of a plasma, because they are usually only partially neutralized by positive ions. Axial magnetic fields between a few hundred and a few thousand gauss are customarily employed to prevent space charge spreading due to the unreliability of positive ion focusing.

We shall first review the types of interactions between electron beams and radiofrequency fields, and discuss the methods of analysis which have proved fruitful. The second part of the chapter comprises a discussion of three experiments on electron plasma oscillations in which the techniques of the microwave tube field have been found to be very useful.

Workers in the microwave tube field have analyzed a large number of very special physical situations, making simplifying assumptions in order to isolate different physical principles leading to various gain mechanisms. Experiments have been performed which verify rather well all of the principles discussed below. The phenomena which have been isolated and studied in detail may also be related to the generally more complex instability problems in other areas of plasma dynamics. For example, the phenomenon of double stream interaction has received a good deal of attention in the astrophysical context, and a number of other interactions that are not as well known but which have nevertheless been studied in considerable detail may find application outside of the electron tube field. Before presenting a survey of these interaction mechanisms, it is appropriate to mention the assumptions that are usually made to simplify the understanding and the analysis of these problems. It is assumed:

(a) that electron velocities are nonrelativistic,
(b) that short-range electron collisions may be neglected,
(c) that perturbations are sufficiently small to permit the use of linearized equations.
(d) that the average drift velocity of the electrons is large compared with the distribution of velocities, and
(e) that in determining the radiofrequency behavior ion motion may be neglected.

Because of their large mass, the radiofrequency motion of the ions is very much smaller than that of the electrons. Thus the ions are not significant in determining the radiofrequency properties of the electron beam, although they may be very important in determining the steady-state or d-c beam conditions. Inclusion of the effects neglected by the above assumptions would ordinarily lead to small corrections on the first-order behavior. In a few situations of great interest one or more of the above assumptions are not met, and these situations have also been studied theoretically and experimentally. However, because of the limitation of space only the first-order effects will be stressed in this survey.

4–2 Interaction mechanisms leading to amplification. The aim of the microwave tube engineer has been to generate and amplify microwave signals. Nearly all of the mechanisms which have been proposed involve a spatial growth in which the signal is larger at a later point on the electron beam than it was at an earlier point. Microwave energy is usually obtained at the expense of the energy in the directed motion of the electron beam. In the discussion below, exceptions to these two generalities are noted as they occur. Figures 4–1 and 4–2 are schematic representations of a number of these interactions. Others have been investigated, but the ones shown illustrate most of the basic types. (The reader should also consult Section 5–9.)

In the klystron interaction [1], a radiofrequency variation in the velocity of electrons is introduced (ideally) at one plane by an input cavity or resonant circuit. The electrons then drift to form bunches of charge in the beam, and these bunches induce current in an output cavity or resonant circuit. This is, in many respects, one of the simplest of the interactions.

The traveling wave interaction [2] shown in Fig. 4–1 is of the usual forward wave type where an input is introduced on a slow wave circuit shown here as the helix, along which a wave travels with a velocity which is very nearly equal to that of the electrons. This is an extension of the klystron interaction in that the bunching effects and the induction effects are distributed uniformly along the beam. This interaction can take place over a very broad band of frequencies provided that these two velocities, that of the circuit and that of the electron beam, are kept nearly equal over the desired band.

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<tr>
<td>Magnetron interaction</td>
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**Fig. 4-1. Summary of electron beam interactions.**

The so-called backward wave interaction [3, 4, 5] illustrated next is a special case of the traveling wave interaction. The interaction is between the electron beam and a circuit wave with phase velocity in the direction of the electron beam and in approximate velocity synchronism with it, as in the forward wave interaction. However, the circuit wave is one in which group velocity and hence the direction of energy flow is in the opposite direction from the phase velocity. A circuit wave with oppositely directed phase and group velocities is termed a backward wave. Although the backward wave interaction is a special case of traveling wave interaction, two interesting differences from the forward wave interaction are important, both of which arise automatically because of the oppositely directed phase and group velocities. (1) As energy is removed from the

<table>
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<td>Inductive wall</td>
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<td>Resistive wall</td>
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<td>Velocity jump</td>
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<td>Rippled wall or rippled stream</td>
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<td>Electron-plasma interactions</td>
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<td>Parametric amplifier (energy from pump)</td>
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**Fig. 4–2.** Summary of electron beam interactions.

electron beam at each point, this energy travels along the circuit in the opposite direction from the electron flow, automatically providing a feedback. Thus backward wave devices are always regenerative, giving oscillation above some minimum current and regenerative amplification for lower values of current. (2) The circuit will be dispersive, that is, its phase velocity will necessarily vary with frequency. This means that the backward wave device does not amplify a broad band of frequencies, but is instead a narrow-band electrically tunable device. Backward wave tubes have been technologically important because of this characteristic.

The double stream interaction [6, 7], a special case of multiple stream interaction, is an interaction between the electrons of one stream and the

electrons of another stream of different average velocity. Energy is transferred from electrons of the fast stream to electrons of the slow stream. The sketch shows the two beams physically separated, but actually it is very important that the two beams be relatively closely coupled. When the transition from two separate velocities to a distribution of velocities is made, spatial growth is obtained only when there is a certain character to the velocity distribution. For growth there must be two relatively well-defined groups of charged particles traveling with different average velocities. This interaction finds application in several branches of physics and astrophysics and has been analyzed in some considerable detail. A more detailed discussion of it is given in a later section.

Next shown in Fig. 4–1 is the magnetron type of interaction [8, 9], although it is an oversimplification to combine all of the various forms of magnetron interaction into one case. The electrons in this type of device drift at right angles to both the electric field $E$ and the magnetic field $B$, with an average velocity $E/B$. In the common forms of this device there is an interaction between the drifting beam of electrons and the slow wave of a circuit which has a phase velocity approximately equal to the electron velocity. The circuit wave may be of either the forward or the backward wave type. One clear difference from the ordinary traveling wave interaction is that the radiofrequency energy comes primarily from the change in potential energy of the electrons as they fall through the d-c electric field. Because the change in kinetic energy of the electrons of the beam may be relatively insignificant, devices of this type are usually more efficient than others in the conversion of d-c energy to radiofrequency energy. This is probably one of the most interesting devices, and is most closely related to some of the plasma problems pertinent to this meeting. Unfortunately, it is also one of the most difficult to analyze and to understand in detail. Electron beams focused by crossed static electric and magnetic fields are inherently unstable. This instability is discussed in a later section.

In Fig. 4–2 there are several examples of rather different kinds of interaction. In the first two there is still interaction between an electron beam and a circuit coupled to the beam, but in both cases the circuit is non-propagating by itself. In the first case, the inductive wall interaction [10, 11], the circuit shows an inductive reactance, but since it is not supplied with a distributed capacitance there is no propagation in the absence of the electron beam. In the resistive wall interaction [12, 13] the circuit is

a pure resistive material. In both situations a growth phenomenon takes place in which energy is removed from the electron beam, with the field quantities and beam perturbations growing exponentially along the beam. A discussion of this interaction from the energy point of view follows in a later section.

In the next two interactions the steady-state or d-c beam conditions are changed periodically along the electron beam. In the velocity jump interaction [14] the d-c velocity of the electron beam is changed periodically by the accelerating and decelerating fields in the gaps. The electron beam is first increased in velocity and then allowed to drift for a specified length, then it is suddenly decreased in velocity and allowed to drift again for the appropriate length (one quarter of a plasma wavelength) before jumping back to the original velocity and repeating the entire cycle. A similar phenomenon occurs when there are periodic variations in any of the d-c or steady-state conditions of the electron beam [15, 16]. The periodic variation may be obtained by varying the diameter of the conducting boundary that surrounds the electron beam, or it may be obtained by a periodic variation in the diameter of the electron beam itself. These two cases are called, respectively, rippled wall and rippled stream amplification. These forms are closely related, and the propagation of disturbances along the electron beam are described by a Matthieu-Hill type of differential equation. The exponentially increasing solutions describe the growing wave phenomenon of interest.

The type of interaction which is probably of most direct interest in the field of plasma dynamics is the electron beam-plasma interaction [17, 18]. Here, the electron beam interacts with the oscillating plasma electrons; the bunches formed in the electron beam tend to excite the plasma oscillations and the plasma oscillations in turn bunch the beam more tightly. This is a generalization of the multiple stream interaction. Disturbances on the electron beam or in the plasma grow spatially in the direction of the electron beam if the frequency of the disturbance is less than the electron plasma frequency. Plasma oscillations excited by this mechanism have been proposed as the source of certain high-intensity bursts of radio noise from the sun [19, 20].

Finally, there is a type of interaction considerably different from the others which have been described, in that the radiofrequency energy that

goes into the electromagnetic fields comes neither from the d-e kinetic
energy of the electrons nor, as in the case of the magnetron, from the d-e
potential energy. Instead, the source of energy is a radiofrequency signal,
a "pump," at a different radiofrequency. When the large amplitude pump
signal is applied to the electron beam (in the simplest case the pump
frequency is twice the frequency to be amplified) the parameters of the
electron beam are caused to vary periodically with the pump frequency.
When the signal frequency propagates along the electron beam whose
parameters vary periodically with time, the signal amplitude increases
exponentially with distance [21]. Energy at the signal frequency is ob-
tained at the expense of energy at the pump frequency. Several forms of
parametric amplifiers have been proposed [22, 23, 24, 25, 26], not all of
which involve electron beams.

Parametric amplification loomed important to microwave technology,
since it appears to make possible very low noise microwave amplifiers.
It is also very interesting to speculate whether parametric excitation of
disturbances may occur in nature in other kinds of plasma regions where
there may be oscillations at several frequencies simultaneously.

While the preceding survey is not inclusive, it does describe briefly the
representative types of interactions with which microwave tube engineers
have been intimately concerned. In the section which follows the methods
of analysis used to describe these processes quantitatively are reviewed.

4-3 Analytical methods. The most successful approach and at the same
time the most straightforward one has been to obtain self-consistent solu-
tions [27, 28] of the combined Maxwell equations for the field, and of the
continuity and momentum equations for the dynamical motion of the
electron "fluid".

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},
\]

\[
\frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t},
\]

   46, 1301 (1958).
\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \quad (4-3) \]
\[ \nabla \cdot \mathbf{B} = 0, \quad (4-4) \]
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \quad (4-5) \]
\[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{e}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (4-6) \]
\[ \mathbf{J} = \rho \mathbf{v}. \quad (4-7) \]

The linearized equations for small perturbations (subscripts 1) are

\[ \nabla \times \mathbf{E}_1 = -\frac{\partial \mathbf{B}_1}{\partial t}, \quad (4-1') \]
\[ \frac{1}{\mu_0} \mathbf{v} \times \mathbf{B}_1 = \mathbf{J}_1 + \varepsilon_0 \frac{\partial \mathbf{E}_1}{\partial t}, \quad (4-2') \]
\[ \nabla \cdot \mathbf{E}_1 = \frac{\rho_1}{\varepsilon_0}, \quad (4-3') \]
\[ \nabla \cdot \mathbf{B}_1 = 0, \quad (4-4') \]
\[ \frac{\partial \rho_1}{\partial t} + \nabla \cdot \mathbf{J}_1 = 0, \quad (4-5') \]
\[ \frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_1 \cdot \nabla) \mathbf{v}_1 + (\mathbf{v}_1 \cdot \nabla) \mathbf{v}_1 = -\frac{e}{m} (\mathbf{E}_1 + \mathbf{v}_1 \times \mathbf{B}_1 + \mathbf{v}_1 \times \mathbf{B}_1). \quad (4-6') \]
\[ \mathbf{J}_1 = \rho_0 \mathbf{v}_1 + \rho_1 \mathbf{v}_1, \quad (4-7') \]

where \( \mathbf{E} - \mathbf{E}_0 \mid \mathbf{E}_1, \mathbf{B} - \mathbf{B}_0 \mid \mathbf{B}_1, \) etc., and terms which are quadratic in the perturbation are neglected. The neglect of the magnetic force term \( \mathbf{v}_0 \times \mathbf{B}_1 \) in comparison with \( \mathbf{E}_1 \) is consistent with the assumption of non-relativistic electrons. It is usually necessary to make additional simplifying assumptions in order to render this approach tractable. When applied to the propagation of one-dimensional disturbances along a drifting beam of electrons this approach is quite simple and leads to a description of space charge waves. Expressions for the alternating components of current and velocity of the electron beam show a wavelike behavior:

\[ \tilde{i} = (\tilde{C}_1 e^{i\beta_1 z} + \tilde{C}_2 e^{i\beta_2 z}) e^{i\omega t}, \quad (4-8) \]
\[ \tilde{v} = -\frac{1}{\rho_0} \frac{\omega_p}{\omega} (\tilde{v}_1 e^{-i\beta_1 z} - \tilde{v}_2 e^{-i\beta_2 z}) e^{i\omega t}, \quad (4-9) \]
where $\beta_1 = (\omega - \omega_p)/v_0$, $\beta_2 = (\omega + \omega_p)/v_0$, $\omega_p = \sqrt{-\rho_0 e/\epsilon_0 m}$ is the electron plasma frequency; $\rho_0 (\rho_0 < 0)$ and $v_0$ are the average charge density and average velocity of the electron beam; and $\mathcal{C}_1$ and $\mathcal{C}_2$ are arbitrary complex constants which can be chosen to meet the initial conditions. The real part of the above expressions for current and velocity is to be taken.

The first of these two wave solutions is called the fast space charge wave, since its phase velocity is slightly greater than the velocity of the electrons. The second is called the slow space charge wave, since the phase velocity is slightly less than the electron velocity. These waves, when viewed by an observer moving with the average velocity of the electrons, appear as electron plasma oscillations with a frequency $\omega_p$. The frequency $\omega$ seen by a stationary observer is simply the Doppler-shifted plasma frequency.

These results apply only to the propagation of disturbances which are independent of the two transverse coordinates. When the beam diameter is comparable to the wavelength of the disturbance the correction for the finite beam diameter becomes important. In this case the frequency of oscillation in a coordinate system moving with the electrons is no longer $\omega_p$; instead it is something less than $\omega_p$,

$$\omega_q = R\omega_p, \quad R = R(\beta b) \leq 1, \quad (4-10)$$

where $\omega_q$ is the reduced plasma frequency and $R$, a function of the beam radius divided by the wavelength of the disturbance, is called the space charge wave reduction factor. Actually, in a beam of finite diameter there is a doubly infinite set of space charge waves, each having a different dependence upon the transverse coordinates. The lowest order mode, in which the current and velocity are nearly constant across the cross section, is the one which is usually of most interest. Figure 4-3 shows how the reduction factor for the lowest circularly symmetric mode depends on the ratio of beam diameter to wavelength. For very short wavelengths the reduction factor is nearly unity, and for very long wavelengths it becomes quite small. The reduction factor arises from the fact that not all of the electric fields originating on bunches of charge stay within the electron beam; instead, some of the field lines bulge outside the beam and some actually terminate on image charges in the metallic conductor which surrounds the electron beam. Figure 4-4 shows the electric field configuration for a short and a long wavelength disturbance. In the case of the short wavelength disturbance the electric field lines emanating from bunches terminate, for the most part, on anti-bunches. In the long wavelength case most of the field lines terminate on the metallic conductors surrounding the electron beam. When the wavelength is long the longi-
**Fig. 4-3.** Plasma frequency reduction factors for a cylindrical beam with infinite magnetic focusing field (lossless, circularly symmetric mode).

**Fig. 4-4.** Electric field configuration for plasma wave in moving reference frame. (a) Short wavelength disturbance; (b) long wavelength disturbance.
tudinal debunching forces are weaker and hence the natural oscillation frequency of the plasma is reduced. Most of the one-dimensional results can be extended to the finite beam case by simply replacing \( \omega_p \) by \( \omega_q \) wherever it appears.

The simplest application of the space charge wave results is to the klystron. The first resonator or tuned circuit introduces a velocity modulation \( \dot{v}(0) \) and no current modulation. To meet these boundary conditions the constants \( \tilde{C}_1 \) and \( \tilde{C}_2 \) in Eqs. (4-8) and (4-9) should be taken as

\[
\tilde{C}_1 = -\tilde{C}_2 = \frac{\rho_0}{2} \frac{\omega}{\omega_p} \dot{v}(0). \tag{4-11}
\]

Equations (4-8) and (4-9) can then be written as

\[
\tilde{v}(z) = j \mu_0 \frac{\omega}{\omega_p} \dot{v}(0) \sin \frac{\omega z}{v_0} e^{j \omega t - z/v_0}, \tag{4-12}
\]

\[
v(z) = v(0) \cos \frac{\omega z}{v_0} e^{j \omega t - z/v_0}. \tag{4-13}
\]

The velocity modulation will decrease in a sinusoidal fashion with distance, and the current modulation will increase in a sinusoidal fashion with distance, as shown in Fig. 4-5. The greatest current modulation of the electron beam is obtained at a distance equal to one-quarter of a reduced plasma wavelength beyond the modulating cavity. The signal induced in the second cavity will be greatest when it is placed at this plane.

An additional one-quarter of a reduced plasma wavelength beyond this plane the space charge debunching forces will have converted the current modulation back to velocity modulation.

From the point of view of the space charge wave analysis, the other devices shown in Figs. 4-1 and 4-2 differ from this example primarily in

Fig. 4-5. Current and velocity modulation in a klystron according to space-charge wave theory.
the type of boundary conditions seen by the electron beam. The velocity
jump interaction can be analyzed by a repeated application of these space
charge wave equations. The traveling wave interaction, on the other
hand, must be analyzed by replacing the metallic conductor which is
assumed to surround the electron beam by the transverse boundary condi-
tions appropriate to the slow wave circuit [29, 30]. The inductive wall
and resistive wall amplifiers can be analyzed in a similar manner [11].

The really interesting result of such an analysis is that the propagation
constants β which are found are not always purely real, but sometimes
show a positive imaginary part. The resulting wave grows exponentially
with distance, and energy is transferred from the electron beam to the
electromagnetic fields continuously along the beam.

The space charge wave type of analysis, while straightforward and very
useful, by itself is limited in its ability to tie all of these possible interactions
together. Two important principles have emerged which have been very
successful in bringing out the relations between the various types of inter-
action. These are the kinetic power theorem [31, 32, 33, 34], which
expresses the power flow along an electron beam associated with the signals,
and the coupling of modes of propagation [35, 30, 37].

It is possible by straightforward manipulations of Eqs. (4–1) through
(4–7) to prove an energy conservation theorem for electron beams [38]:

\[
\frac{\partial}{\partial t} \left[ -\frac{m}{2e} \rho \nu^2 + \frac{1}{2} \rho \nu^2 + \frac{1}{2c_0^2} \nu^2 + \mu_0 \nu^2 \right] + \nabla \left[ -\frac{m}{2e} \rho \nu^2 \nu + \mathbf{E} \times \mathbf{H} \right] = 0.
\]

(4–14)

We recognize this as simply an extension of Poynting's theorem in which
there now appears a term \(-m \rho \nu^2 / 2e\) which represents the kinetic energy
density of the electrons and a term \(-m \rho \nu^2 \nu / 2e\) which represents the
kinetic power flow of the electron beam. With this theorem it is possible
to calculate the energy density and flux when the field quantities and the
fluid motions are known. There is, however, an essential difficulty in the
application of this theorem. The field quantities, the fluid velocity, and

the density are calculated only to first order in the amplitude of the disturbance, whereas it is necessary to calculate the energy density and flux to second order in the amplitude of a disturbance if we are to obtain the time average energy density and flux associated with the disturbance. To calculate the energy density and flux correctly to second order the field quantities and the electron fluid quantities must also be known correctly to second order. This approach has actually been carried through for a few special cases [39] but it is not generally useful, since the solution to second order in the amplitude of the disturbance must be found.

An alternative and very fruitful approach to the problem of calculating energy densities and fluxes has been to prove conservation theorems completely within the framework of the linearized equations (4-1') through (4-7'). It is possible, for example, by manipulating these equations in a manner similar to that used in deriving the Poynting theorem, to show that

$$\frac{\partial}{\partial t} \left[ - \frac{m}{2e} \left( \rho_0 \rho_1^2 + 2 \rho_0 \rho_1 \rho_1 \right) + \frac{1}{2} \rho_0 \rho_1^2 + \frac{1}{2} \mu_0 H_1^2 \right]$$

$$+ \nabla \cdot \left[ - \frac{m}{2e} \rho_0 \rho_1 \mathbf{j}_1 + \mathbf{E}_1 \times \mathbf{H}_1 \right] = 0 \quad (4-15)$$

provided only that the perturbation in electron velocity is in the same direction as the drift velocity (as will be the case for a strong magnetic containing field). This result is quite similar to the extension of Poynting's theorem, Eq. (4-1'), except that each term is quadratic in the amplitude of the disturbance. Since this equation is in the form of an energy conservation law the quantities in brackets are usually referred to as the energy density and the energy flux, respectively, associated with the disturbance. This interpretation is supported by calculations for special cases in which the time average power flow given by Eq. (4-15) is in agreement with the result of Eq. (4-14) when solutions are known to second order in the amplitude of the disturbance. The quantity $\mathbf{E}_1 \times \mathbf{H}_1$ is the electromagnetic power flow associated with the disturbance, and the remaining quantity in the brackets is the kinetic power flow associated with the disturbance. The quantity $-(m/e) \nu_0 \rho_1$ is called the kinetic voltage, $U_k$. The time average power flow across a plane of a cylindrical system in which the disturbances are sinusoidal functions of time is given by

$$P = \text{Re} \int_{\text{cross section}} \left[ (\tilde{U}_1 \mathbf{j}_1 + \tilde{E}_1 \times \tilde{H}_1) \cdot d\mathbf{S} \right]. \quad (4-16)$$

It is important to note that the kinetic power flow here is that associated

with the disturbance itself; that is, it is the difference between the total kinetic power flow when a disturbance is present and the kinetic power flow when the disturbance is absent. A very interesting consequence of this definition is that it is possible for the kinetic power flow defined in this way to be negative (power flow in a direction opposite to that of the electron motion), whereas the total power flow must always be positive. For the a-c kinetic power flow to be negative the disturbance must be such that the perturbation in velocity is negative when the perturbation in the number of particles passing a plane is positive, signifying that more particles of reduced velocity pass the plane than do particles of increased velocity. The time average energy passing the plane is then decreased.

When this result is applied to the space charge wave solutions discussed previously, it is found that the kinetic power flow associated with the fast space charge wave is positive, but that the kinetic power flow associated with the slow space charge wave is negative. The negative energy of the slow space charge wave is very important, since this means that it is possible to remove energy from the electron beam by increasing the amplitude of this mode of propagation. Similarly, increasing the amplitude of the fast space charge wave requires the addition of energy to the electron beam. This result, together with an elementary theory of the coupling of two modes of propagation, can be used to explain the operation of most microwave electron tubes.

The theory of coupling of modes of propagation in a system which supports guided waves is quite closely related to the coupling of modes of oscillation of a vibrating system and deals with two modes of wave propagation which are assumed to be weakly coupled. When the modes are coupled continuously along the length of the system, differential equations for the mode amplitudes \( P \) and \( Q \) may be written [37]:

\[
\frac{\partial \ddot{P}}{\partial x} + j\beta P \dot{P} - \kappa \dot{Q} = 0, \tag{4-17}
\]

\[
\pm \kappa \ddot{P} + \frac{\partial \ddot{Q}}{\partial x} + j\beta \dot{Q} = 0. \tag{4-18}
\]

The mode amplitudes \( P \) and \( Q \) are defined so that their absolute square gives the power flow associated with each mode. The upper sign in Eq. (4-18) is to be taken when the power flows of the two modes are in opposite directions; the lower sign is to be taken when the power flows are in the same direction. \( \kappa \) is the coupling constant [40]:

\[
\kappa = \frac{\langle \partial \ddot{x} \rangle \langle \dot{P} \dot{P} \rangle}{2 \Re \langle P \dot{Q} \rangle}. \tag{4-19}
\]

40. A. YARIY, unpublished.
When $\kappa$ is zero and there is no coupling between the modes, each propagates with its own unperturbed phase constant:

$$P = P_0 e^{-j\beta_p z}, \quad Q = Q_0 e^{-j\beta_q z}$$

(time dependence of $e^{j\omega t}$ is understood). When the two modes are coupled together the propagation constants of the characteristic waves of the coupled system are given by

$$\beta_1 = \frac{\beta_p + \beta_q}{2} + \sqrt{\frac{\left(\beta_p - \beta_q\right)^2}{2} \pm \kappa^2},$$

$$\beta_2 = \frac{\beta_p + \beta_q}{2} - \sqrt{\frac{\left(\beta_p - \beta_q\right)^2}{2} \pm \kappa^2}.$$  \hspace{1cm} (4-21)

Most interesting is the case where both waves have the same unperturbed phase velocity, $\beta_p = \beta_q$. Then Eq. (4-21) becomes

$$\beta_1 = \beta_p + \sqrt{\pm \kappa^2}, \quad \beta_2 = \beta_p - \sqrt{\pm \kappa^2}.$$ \hspace{1cm} (4-22)

If the power flow of the two uncoupled waves is in the same direction the upper sign is to be taken and the propagation constants of the coupled system are both real but differ slightly. If, however, the power flows of the uncoupled modes are in opposite directions, the lower sign is to be taken and the propagation constants become complex, one having a positive imaginary part and the other a negative imaginary part. Thus one wave increases exponentially with distance and the other decreases exponentially with distance. In the case of increasing or decreasing waves the total power flow of the system must be zero, for only in this way can the power flow be independent of the $z$ coordinate. The power flows of the two modes are equal and opposite at every point along the system.

Through an accounting of power flow at each point along the tube these two principles, i.e., the kinetic power theorem and the theory of coupling of modes of propagation, have led to a much better understanding of most of the other interactions described earlier [36, 37].

The ordinary traveling wave interaction can be understood in terms of a coupling between the forward circuit wave and the slow space charge wave of the electron beam [36]. Circuit power is in the direction of the electron beam, but the slow space charge wave power flow is in the opposite direction from the electron beam, since the kinetic power flow of the slow space charge wave is negative. When these two modes are coupled a growing and decaying pair of waves results, and the growing wave is of interest for amplification. As the disturbances grow exponentially along the device the circuit power increases, the beam power becomes more negative, and
the two balance at each plane, in accord with the power theorem (Fig. 4–6). The significance of the increasing negative power of the electron beam is that energy is being removed from the beam, so that at later points along the device the electrons carry less energy than at earlier points.

In the backward wave interaction circuit power flow is in a direction opposite to that of the electron beam, as is the power carried by the slow space charge wave. Thus both waves carry energy in the same direction.
and no increasing or decreasing waves result [37]. There is instead an interference or beating phenomenon in which energy may be transferred from one mode to the other periodically along the coupled system. In this case the kinetic power of the slow space charge wave becomes more negative and the circuit power flow becomes less negative, as shown in Fig. 4–6. The a-c power flow of the system is in a direction opposite to that of the electron beam and is carried by the electron beam at the point where the beam emerges from the interaction region, and by the circuit at the point where the electron beam enters the interaction region. Power output of this device is taken from the point along the circuit at which the electrons enter.

In the double stream interaction it is the coupling between the fast space charge wave of the slow electron beam and the slow space charge wave of the fast electron beam which gives rise to an exponentially growing wave [36]. The first mode has positive kinetic power and the second mode has negative kinetic power. As the disturbance increases along the electron beam these two kinetic powers exactly balance. Energy is transferred from the fast electron beam to the slow electron beam. Only when the electron beams are allowed to pass through an output coupling circuit is any energy extracted from the two beams.

The resistance wall interaction is an example of the coupling of one propagating mode, the slow space charge mode, and a dissipative circuit. As the disturbance grows larger the power flow of the slow space charge wave becomes more negative. The energy which is removed from the electrons is dissipated in the wall. This principle can be used to make a microwave amplifier even though the resistive wall decreases the total energy of the system, because the amplitude of the disturbance may grow exponentially as a result. This point of view makes it clear that a resistance wall cannot be used to reduce the noise of an electron beam by absorbing this noise, since by removing noise power the noise amplitude is increased rather than decreased. Although this interaction found no important technological applications, it is exceedingly interesting for the understanding it imparts.

From the coupled mode point of view the velocity jump interaction is one in which the fast space charge wave is coupled to the slow space charge wave of the same electron beam periodically. As the signal amplitude grows, the slow and fast space charge wave amplitudes increase together, so that the net power flow of the system remains zero. This interaction is similar to the operation of a periodic electric circuit in its stop band, where both increasing and decreasing waves appear. In that case also energy in one direction is balanced by the reflected energy in the other direction.

This brief discussion was intended to illustrate the usefulness of the power theorem and the coupling of modes of propagation and to describe
qualitatively the interactions which occur in microwave electron tubes. These two principles have contributed greatly to the unification of the theory of many apparently quite different electron devices.

4-4 Periodic circuits and periodic electron beams. Most of the slow wave circuits employed in microwave electron tubes are periodic in one spatial coordinate. Figure 4-7 shows the most common slow wave circuit, a tape helix, which is periodic with a period corresponding to the pitch $p$. Also shown there are the propagation characteristics in the form of an $\omega-\beta$ diagram. Since the phase velocity of the wave is $\omega/\beta$, the slope of a line from the origin to a point on the curve is equal to the phase velocity of the wave at a frequency corresponding to that point. The slope of the curve at that point, $d\omega/d\beta$, is the group velocity at that frequency. If these two quantities have the same sign, the wave is called a forward wave. If, however, they are of opposite sign, the wave is called a backward wave. Although the detailed analysis [41] of the tape helix is rather complicated, the qualitative features of this diagram may be explained in terms of three relatively simple principles [42].

(1) The electromagnetic wave propagates along the tape at approximately the velocity of light, hence the axial phase velocity is the velocity of light times the sine of the pitch angle:

\[ \beta_0 \approx \frac{\omega}{c \sin \psi}, \quad \psi = \text{helix pitch angle}. \] (4–23)

(2) The dependence of any field component upon the axial coordinate may be written [43] as

\[
\begin{bmatrix} E_z \\ H_z \end{bmatrix} = e^{-\beta_0 z} f(z), \quad f(z + p) = f(z),
\]

\[ = e^{-\beta_0 z} \sum_{m=-\infty}^{\infty} C_m e^{i(2\pi m z / p)}. \] (4–24)

where \( f(z) \) is a periodic function of period \( p \) and hence may be expanded in a Fourier series. This is an extension of Floquet's Theorem or, as it is called in solid state physics, Bloch's Theorem. The infinite series can be regarded as a summation of an infinite number of partial waves or, as they are generally called, spatial harmonics, each having a different propagation constant

\[ \beta_m = \beta_0 - \frac{2\pi m}{p} \] (4–25)

and corresponding phase velocity

\[ \nu_m = \frac{\omega}{\beta_m} = \frac{\omega}{\beta_0 - (2\pi m / p)}. \] (4–26)

(3) Propagation in the shaded areas or forbidden regions is not possible because one of the spatial harmonics would then have a velocity greater than the velocity of light. Only when the velocities of the spatial harmonics are less than the velocity of light does the field decrease sufficiently rapidly with radius for the energy of the system to be bounded.

In Fig. 47 the curve labeled \( m = 0 \) is the principal spatial harmonic, or fundamental. Other spatial harmonics are shown in a limited part of the diagram. Generally speaking, the spatial harmonic with the smallest propagation constant has the largest amplitude and spatial harmonic amplitudes decrease with increasing values of \( m \). In periodic circuits, therefore, there are many spatial harmonics which might be used, some of which have phase and group velocities in opposite directions. It is the problem of the tube designer to find spatial harmonics with sufficiently

strong fields per unit power for proper interaction over the band of frequencies of interest.

Less common than the concept of periodic circuits is the concept that in periodic beam devices (two or three examples were shown in Figs. 4–1 and 4–2) the periodic nature of the electron beam gives rise to spatial harmonics of the slow and fast space charge waves. Simply applying Eq. (4–25) to the space charge wave propagation constants leads to the $\omega$-$\beta$ characteristics of a beam, as shown in Fig. 4–8. In addition to the usual slow and fast space charge waves labeled “beam fundamental” there are the various spatial harmonics of these waves, indicated as $-1$ beam harmonic, $+1$ beam harmonic, and others. Also shown in the figure are the characteristics of a folded transmission line type of slow wave circuit. At the frequency corresponding to each intersection of the circuit curve and a space charge wave curve, traveling wave interaction is possible. Interactions involving spatial harmonics of the slow space charge wave are of primary interest, since energy may be extracted from the electron beam. At frequencies $\omega_1$, $\omega_3$, and others, interactions between spatial harmonics of the slow space charge wave and a backward spatial harmonic of the circuit wave are possible. At frequencies $\omega_2$, $\omega_4$, and others, interactions between spatial...
harmonics of the slow space charge wave and a forward spatial harmonic of a circuit mode are possible.

One interesting application of spatial harmonics of a periodic electron beam is the possible interaction with fast wave circuits. In the conventional traveling wave interaction with a nonperiodic electron beam it is necessary to slow the wave velocity down to the electron velocity. However, when the beam is periodic some of the spatial harmonics of the space charge waves have a very high phase velocity and are therefore capable of interacting with a high-velocity circuit, perhaps a smooth waveguide in which the phase velocity is greater than the velocity of light. The characteristics of such a waveguide are shown in Fig. 4–8 by the dotted line. At frequency ω5 interaction is possible between the —3 spatial harmonic of the slow space charge wave and the smooth waveguide. However, this interaction will be quite weak, since the —3 spatial harmonic of a slow space charge wave is much weaker than the fundamental spatial harmonic.

### 4–5 Interactions of electron beams in crossed electric and magnetic fields.

The simplest type of magnetron interaction is one in which the electron beam is a thin sheet and individual electron trajectories are straight lines, the electron velocity being E/B. Spatially growing waves are possible in such a beam in the absence of a slow wave circuit and the absence of a second beam [44, 45]. The growth mechanism is sufficiently interesting and simple to warrant further explanation. The phenomenon appears simplest in a coordinate system moving with the average velocity of the electrons. In this coordinate system there is no static electric field and the electrons are at rest in the absence of a perturbation. The magnetic field is still present. Consider the initial sinusoidal displacement of the strip beam shown in Fig. 4–9. The electrostatic force on electrons at point a is upward. To see this, note that the longitudinal component of force from electrons equidistant from a but on either side exactly cancel, whereas the transverse components add to produce the upward force. The total force on a point a is the sum of forces from all such pairs of electrons on either side of it. Were it not for the magnetic field it would be immediately clear that such a perturbation would increase with time. The presence of a strong magnetic field reduces the rate of growth of the disturbance but does not eliminate it. Because of the strong magnetic field the electrons will drift at right angles to the electric force and to the magnetic field. This motion, indicated by the arrows in Fig. 4–9, results in an accumulation of charge at c. The space charge debunching force which results from this

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accumulation of charge at c is longitudinal and, together with the magnetic field, it causes a transverse drift in the direction of the original force, in such a way as to increase the original perturbation. The growth of these disturbances is exponential in time (i.e., $e^{\gamma t}$). The rate of growth $\gamma$ is given by

$$\gamma = \frac{\sigma_0}{2\varepsilon_0 B} \beta,$$

(4-27)

where $\sigma_0$ is the charge per unit area, $B$ is the magnetic field strength, and $\beta$ is $2\pi$ divided by the wavelength of the disturbance.

When this instability is viewed in the laboratory frame of reference the disturbances are transported with the drift velocity of the electrons $v_0$, and they appear to grow exponentially with distance with the spatial rate

$$\alpha = \gamma v_0.$$

(4-28)

Equation (4-27) predicts that the rate of growth becomes very large for short wavelength disturbances. The analysis from which (4-27) was obtained omits consideration of the beam thickness, and this result can be shown to apply only so long as the wavelength of the disturbance is much greater than the beam thickness. Analysis of the thick beam case requires a more detailed consideration of the electron trajectories in a thick beam. Assuming laminar flow, it is readily seen that all electrons cannot have the same drift velocity $v_0$, since the transverse electric field is different in different parts of the beam owing to the presence of the charge. Electrons at the lower edge of the beam travel somewhat more slowly than electrons at the upper edge of the beam. Self-consistent field solutions for small perturbations have been obtained by the methods described earlier for this type of flow [46, 47]. The dependence of the rate of growth upon wavelength is given in Fig. 4-10. For long wavelengths the result is the

47. R. W. Gould, unpublished.
Fig. 4-10. Spatial rate of growth of a beam of thickness $t$ in crossed electric and magnetic fields.

The same as (4–97). For wavelengths about eight times the beam thickness the rate of growth is largest.

Hollow electron beams focused by an axial magnetic field exhibit a similar instability [48, 49]. The explanation outlined in Fig. 4-9 is directly applicable with two minor modifications: (1) the electron beam is not a strip but is folded around on itself, and (2) the electrons drift along the magnetic field lines. The spatial rate of growth along the axis of the electron beam is given by Eq. (4–28) if $v_0$ is interpreted as being the drift velocity along the electric field lines. The curvature of the cylindrical beam is important only if the wavelength is comparable to the radius of the hollow beam. If it is assumed that disturbances of all wavelengths are excited to an equal extent initially then the wavelength with the highest rate of growth is the one observed at a later point along the beam. For hollow beams which are thin in comparison with their radius the theory of the planar beam is applicable, and disturbances whose wavelength along the circumference of the beam is eight to ten times the beam thickness should be observed. However, periodic deviations in electron emission or asymmetries in the electron gun structure may cause disturbances of certain wavelengths to be preferentially excited initially.

The magnetron type of interaction has been successfully employed to make high efficiency (25–40%) voltage tunable backward wave oscillators [50]. The space charge forces from electron-electron interactions, which give rise to the instability just described, can cause a marked decrease in the starting current of the backward wave oscillator, or an increase in gain of a forward wave amplifier [51]. It seems likely that the

electron-electron interaction of this type is responsible for the interchange of energy between electrons which is necessary to account for current to electrodes that are negative with respect to the cathode [44]. Without such an interchange no electrons should reach negative electrodes. The thin beam model and laminar flow beam model represent idealizations which cannot be achieved in practice.

Figure 4.11 shows schematically a typical geometry with more nearly typical electron trajectories. At each point in the beam there are electrons traveling in different directions. Some progress has been made in the analysis of this situation through the introduction of a pressure term in the equation of motion [52]. This problem and the more complicated situation in the magnetron itself still present formidable analytical difficulties. Yet another device in which multivelocity flow is encountered is the re-entrant beam tube shown in Fig. 4.12. Electrons emitted from the cathode make several transits through the interaction region before striking one of the electrodes. Electrons do not follow the same trajectory on successive transits. Extensive tests have been made on the steady-state electron trajectories [53] in order to determine a beam model with which to study radiotransmission perturbations. The position and density of the electron beam can be determined experimentally by introducing a small amount of gas which is ionized by the electron beam. Visible recombination radiation gives qualitative information about the location and density of the beam.

4-6 Velocity distribution effects in electron beams. In analyzing the interactions described here it is usually a good approximation to assume that the electron velocity is a single-valued function of position and time. There are, as already noted, a few situations in which this is not a valid approximation and velocity distribution effects are important. Near the cathode and potential minimum in front of the cathode the spread in

electron velocities is comparable to the average velocity. It is no longer accurate to analyze the propagation of disturbances in this region assuming a single-velocity beam. When large-amplitude disturbances are present on an otherwise single-velocity stream there can be overtaking of one group of electrons by another, leading to electrons of several different velocities at a given point in space and time. Since the number of different velocity classes is generally small, it is usually convenient to follow each class of electrons (the Lagrangian approach). Finally, multivelocity effects are important in thermal plasmas.
The most extensive investigation of multivelocity effects in electron beams has been in connection with the propagation of noise near the potential minimum [54, 55]. Because of the difficulty of the problem all analyses have been one-dimensional. The analytical approaches fall into three general classes:

(a) Integration of the one-dimensional Boltzmann equation, neglecting short-range collisions.

(b) The use of the hydrodynamic equations including a pressure term, neglecting heat flow (the adiabatic approximation) [56].

(c) The use of a separate equation of motion for each electron or for each class of electrons, and using the total charge density and current density produced by all electrons or class of electrons in the Maxwell equations.

The problem to which these methods have been applied can be described as follows: Electrons are emitted from the cathode at random times with a distribution of velocities which is nearly half Maxwellian. Because of the discrete nature of the electronic charge and the random times of emission, there is a fluctuation in the amount of current emitted in each velocity class, a shot noise. The problem is then to calculate the corresponding fluctuation in total current and in the average velocity at a later point in the electron beam. It is sufficient to make this calculation at a point along the beam where the spread in velocities is small compared with the average velocity, for beyond that point the single-velocity theory is adequate. A considerable advance in the understanding of noise in microwave tubes has been made by attempting to describe propagation noise from the cathode through the tube entirely in terms of the single-velocity theory [57, 58, 59]. Such a theory has as its input conditions at the cathode the shot noise fluctuation in current and a corresponding fluctuation in the average velocity of the emitted electrons [60]. It can be shown in quite a general way from this approximate theory that the minimum noise factor which can be obtained with conventional microwave tubes should be about four (6 db) [58]. In order to achieve this minimum noise figure, it is necessary to accelerate the electron beam to its final velocity in a very special way. This may be accomplished by shaping the voltage profile along the axis of the electron beam through the use of cylindrical or disk-

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type electrodes operated at different potentials, \( V_1, V_2, \ldots \), as shown in Fig. 4-13. Recently, noise factors of about 2.5 have been obtained, thus pointing up the inadequacy of this theory. The principal shortcoming is thought to be the use of the single-velocity approximation near the potential minimum. Numerical solutions of the Döpker-schum equation have been carried out for the low-velocity region and it is found that a correlation between velocity and current fluctuations can develop, thus allowing a lower noise factor than predicted by the single-velocity theory [55]. It is still too early to say how general these results are; this problem is still being actively investigated.

The other situation in which velocity distribution effects are important and which has received considerable attention is the multiple-stream interaction. Assuming sinusoidal time and space dependence, \( e^{j(\omega t - \beta z)} \), it is easily shown that the propagation constant, \( \beta \), for one-dimensional disturbances in an \( N \)-beam system must satisfy the equation

\[
1 = \sum_{i=1}^{N} \frac{\omega_{pi}^2}{(\omega - \beta v_0 i)^2},
\]

where \( \omega_{pi} \) is the plasma frequency of the \( i \)th stream and \( v_0 i \) is the drift velocity of the \( i \)th stream in the direction of propagation. Solutions for the two-beam case have been studied extensively [61] and have been subjected to direct experimental verification [6]. When there is a continuous distribution of drift velocities the summation in equation (4-29) is replaced by

an integration [62]. A more common approach has been to use the linearized Boltzmann equation

$$\frac{\partial f_1}{\partial t} + (v \cdot \nabla)f_1 + \frac{e}{m} E_1 \cdot \nabla_E f_0 = \left( \frac{\partial f_1}{\partial t} \right)_{\text{coll}},$$  \hspace{1cm} (4-30)

where \( f_1 \) is the perturbation from the equilibrium distribution \( f_0 \) (see Section 5-9). Long-range Coulomb forces are included in the electric field \( E_1 \) and short-range collisions are usually neglected or approximated by a relaxation term

$$\left( \frac{\partial f_1}{\partial t} \right)_{\text{coll}} = -\nu_c f_1,$$  \hspace{1cm} (4-31)

where \( \nu_c \) is an average collision frequency. This approach leads to a dispersion relation

$$1 = -\frac{\omega^2}{k} \int f_0(v) \frac{dv}{\omega}$$  \hspace{1cm} (4-32)

which can be shown to be equivalent to Eq. (4-30). When particles of all velocities are present, a fundamental difficulty arises in the application of Eq. (4-31), since the denominator vanishes for particle velocity equal to the wave velocity. In solving the problem by the Laplace transform method, Landau has shown that the integration in Eq. (4-32) should be interpreted as an integration in the complex velocity plane [62]. The resulting plasma oscillations show a damping which is negligible for very long wavelengths but which increases as the wavelength becomes shorter. The damping is intimately related to the behavior of particles traveling with velocities approaching the wave velocity.

To accommodate the very large freedom in initial conditions, other types of solutions, representing a convection of the disturbance by particles of a particular velocity, are also possible [62, 63]. Furthermore, if one admits singular initial distributions, wave solutions of arbitrary frequency and damping wavelength are possible [64, 65].

It is always necessary to verify that these solutions fulfill the conditions for linearity, especially if some particles have a velocity nearly equal to the wave velocity. While it is difficult to obtain general solutions of the nonlinear Boltzmann equation, a number of traveling wave solutions have been obtained [66].

The transition from a system of $N$ beams to one in which there is a continuous distribution of velocity has been studied in detail [66]. A large number of beams can be superposed to form a Maxwell distribution. Since it is known that the first system is unstable and yet the Maxwell distribution must be stable, the question then arises, how is the resulting superposition stable? The analysis answers the question in the following way. The beams were taken to be equally spaced in velocity but the electron density was allowed to vary from beam to beam. The approximation to a continuous distribution was made by allowing the number of beams, $N$, to approach infinity, while the velocity spacing between the beams, $\Delta v$, and the density of electrons in each beam go to zero at $1/N$. The beams are found to always have unstable modes; however, the growth rates of most of the modes go to zero as $1/N \log N$. If all the unstable modes vanish in this way the distribution is said to be stable. If some modes have finite growth rates in the limit the distribution is unstable. The conditions obtained for stability in this way, of course, agree with the results of the Boltzmann equation.

The method provides a new physical insight into the mechanism of Landau damping or phase mixing. It has also been applied to calculate Landau damping of hydromagnetic waves.

4-7 Nonlinear effects in electron beam dynamics. For many applications, a knowledge of the small-signal behavior of microwave tubes is sufficient. In determining the maximum power output, however, nonlinear effects are of paramount importance. The large-signal theory of klystrons is simple if electron interaction effects can be neglected, since no forces act on the electron in the drift region [1]. In the traveling wave interaction the analysis including space-charge forces is considerably more difficult. The most successful analyses of the nonlinear behavior of microwave tubes have employed high-speed digital computers. The usual approach is to solve the nonlinear equations of motion of a finite number of electrons entering the tube at equally spaced time intervals. Each field of charge interacts with all other units of charge through the coulomb field as well as with a transmission line which represents the slow wave circuit [67, 68, 60, 70]. Such an analysis seems to predict correctly the general behavior in the large-signal regime.

Another approach to analyses of large-signal operation has been to work with the moments of the velocity distribution in order to describe the

multiovelocity situation that exists when there is overtaking [71]. This method, which would also require the use of a computing machine to obtain useful numerical results, has not found much use. Still another method, which is quite satisfactory for the analysis of quasi-small-signal behavior of frequency converters and cross modulation in linear amplifiers, is a method of successive approximations which amounts to an expansion in powers of the signal amplitude [72, 73]. The first few terms describe the principal intermodulation products.

4–8 Interaction between an electron beam and a plasma. The remainder of this chapter is devoted to a description of three experiments dealing with electron plasma oscillations performed during the past year or two at the California Institute of Technology. These experiments utilize some of the concepts developed in the microwave tube field, although they actually involve plasmas derived from a low-pressure mercury arc discharge. Electron densities are in the range $10^8$ to $10^{12}$ per cubic centimeter, and hence the electron plasma frequencies are in the low microwave frequencies. The electron collision frequency is usually considerably less than either the operating frequency or the electron plasma frequency. The plasmas are approximately electrically neutral and, with the exception of the third experiment, no static magnetic fields are involved. At microwave frequencies the ion motion can be neglected.

Numerous theoretical analyses have been made which predict spatially growing waves when a directed beam of electrons traverses a plasma. (This problem is also discussed by James Tuck in Section 1–5 and by William Allis in Section 3–4.) It has been suggested that the excitation of large-amplitude plasma oscillations by this mechanism may be responsible for high-intensity bursts of solar radio noise which has its origin in the solar corona [19]. Attempts to verify the existence of spatially growing waves in laboratory experiments have previously been unsuccessful [74, 75]. The spontaneous oscillations observed in these experiments seem to be associated with sheath oscillations. The experiment described below represents a different approach in that the electron beam is first modulated with the microwave signal before it passes through the plasma [76]. The modulation of the beam is examined after it emerges from the plasma.

72. O. Doehler and W. Kleen, Ann. radioelec. compagn. franç. assoc. T.S.F.
3, 124 (1948).
73. P. Paczkowski, RCA Rev. 18, 221 (1957).
Figure 4-14 is a schematic drawing of the experimental tube. An arc discharge exists between the cathode and the arc collector in such a way that the plasma interaction region is part of the positive column. A 400-volt beam of electrons generated by the electron gun passes through the plasma interaction region after having been modulated with a radio-frequency signal by the modulation helix. This short helix, by the traveling wave interaction mechanism described earlier, produces primarily a velocity modulation of the electron beam. As the electrons drift, bunches form and the electric field of the bunches sets the plasma into oscillation. The fields of the oscillating plasma, particularly strong if the applied frequency is nearly equal to the electron plasma frequency, produce a further velocity modulation and current modulation. The bunched electron beam induces a signal on the demodulation helix by means of a traveling wave interaction and this signal appears in the output waveguide.

The theory of this experiment dates back to an early paper by Pierce on ion oscillations [77]. Other papers have extended this theory and have

applied it to different problems. The distribution of electron velocities is shown in Fig. 4–15. If the spread and velocities of the plasma and electron collisions are neglected, Eq. (4–29) with \( N = 2 \) and \( v_{01} = 0 \) applies. This equation may be solved immediately for the propagation constant

\[
\beta = \frac{\omega}{v_{01}} \pm \frac{\omega_{p1}}{v_{01} \sqrt{1 - \left( \frac{\omega_{p1}}{\omega_{e2} / \omega_{e1}} \right)}}. \tag{4–33}
\]

The subscript 1 applies to the beam and the subscript 2 to the stationary plasma. When the applied frequency is less than the plasma frequency of the stationary particles, \( \omega_{e2} \), \( \beta \) can have a positive imaginary part standing for a spatially growing wave. The rate of growth increases with frequency,
becoming infinite at \( \omega = \omega_{p2} \). The effect of collisions and random velocities of the thermal plasma is to make the rate of growth finite when \( \omega = \omega_{p2} \), and some growth possible for frequencies slightly larger than \( \omega_{p2} \). A typical result for this experiment is shown in Fig. 4–16, where the spatial rate of growth is plotted versus the square of the plasma frequency for a fixed operating frequency \( \omega \). If the operating frequency is increased the entire curve is shifted to the right.

The experiment is performed by applying microwave input at frequency \( \omega \), varying the discharge current so as to pass through plasma resonance, and observing the power output. When the power output is displayed on an oscilloscope versus discharge current traces of the type illustrated in Fig. 4–17 are obtained. The large increase in output signal occurs at plasma resonance. The results obtained are in qualitative agreement with theoretical predictions of the type shown in Fig. 4–16. At plasma resonance the output is typically 30 to 40 decibels larger than obtained with the discharge turned off. With the discharge off the device behaves as a poorly designed klystron, with the signal being transmitted from the modulation helix to the demodulation helix by the constant amplitude space-charge waves.

![Graph showing output versus discharge current](image)

**Fig. 4–17.** Output of interaction tube versus discharge current with constant input.

If the applied frequency is increased a larger value of discharge current is required to secure plasma resonance. Experimentally it is found that the required discharge current increases as the square of the applied frequency. This result is indicated in Fig. 4–18. In a low-pressure arc discharge of the type used in this experiment the electron density is approximately proportional to the discharge current. Hence the square of electron plasma frequency is proportional to the discharge current. To determine that the maximum interaction actually occurs when the plasma frequency is equal to the applied frequency, electron density in the interaction region
4.9 Scattering of electromagnetic waves from an ionized column. The following experiment was devised to measure electron density in cylindrical columns similar to those used in the electron beam-plasma interaction experiment. The general features of scattering from a cylindrical ionized column have been worked out in connection with the study of reflections from meteor trails [78]. Several laboratory experiments have been performed [79, 80]. The method of density measurements makes use of a transverse resonance of the plasma column. The nature of this resonance is

illustrated in Fig. 4–19. When the electron gas is displaced from the ions by rigid-body displacement as shown, the resulting surface charges generate an electric field which tends to return the electrons to their original position. Upon returning to their original position the electrons have a velocity which causes them to overshoot the equilibrium until the electric force from the surface charges causes them to reverse direction and return toward their equilibrium position. Thus there is a natural mode of vibration of the electrons relative to the ions. The frequency of this oscillation, assuming that there are no variations of the disturbance in the axial direction, is \( \omega_e/\sqrt{2} \). When the plasma is contained by a glass tube, a dielectric correction must be made. This results in a somewhat lower resonant frequency.

The cylindrical plasma column is placed in a rectangular waveguide propagating a TE_{10} mode where the electric field of the waveguide excites this mode of oscillation. If the frequency of the incident wave is equal to the resonant frequency, very large oscillations are obtained, which results in a very large reflected wave. In the experiment the reflection coefficient is measured as a function of discharge current at a fixed frequency. Figure 4–20 shows a typical result. The large peak corresponds to plasma resonance and the reflection coefficient is very nearly equal to unity.

When the incident frequency is increased the curve in Fig. 4–20 is shifted to the right. The discharge current for maximum scattering should be proportional to the square of the incident frequency.

The smaller peak is as yet unexplained. In addition to the dipole mode just described, there are oscillations corresponding to higher-order multipoles. All have the same resonant frequency as the dipole mode and hence cannot be related to the minor peak. In all of these modes there is no charge accumulation in the interior of the plasma. There also exist plasma
oscillations with frequency $\omega_p$, and in these modes there is charge accumulation in the interior of the plasma but no fields external to the plasma. Because there are no external fields these oscillations are not coupled to the waveguide. If it were possible for these oscillations to couple slightly to waveguide fields, this might explain the smaller of the two peaks.

The scattering method has been used to measure electron density or plasma frequency as a function of discharge current. The result is shown in Fig. 4–21, where it may be seen that the square of the plasma frequency
is indeed approximately proportional to the discharge current. The circles in Fig. 4-21 indicate plasma frequency measurements made by the cavity method. The line $K$ is the semiemirical result of Klarfeld. The crosses $W$ were made by the wave propagation method to be described later. The results from the various methods are seen to be in general agreement.

From the explanation given in connection with Fig. 4-19 it appears that the exact resonant frequency may be sensitive to variations of electron density with radius. When the electrons are given a rigid body displacement, a volume charge as well as a surface charge is produced. In the discharge the electron density is known to be somewhat higher in the center of the tube than near the sheath, since recombination takes place primarily at the walls. To evaluate the change in resonant frequency, scattering from a nonuniform column has been studied, assuming that the density varies as

$$\rho = \rho_0 \left(1 - \alpha \frac{r^2}{a^2}\right), \quad (4-34)$$

where $a$ is the radius of the tube and $\alpha$ is a parameter. So long as $\alpha$ is less than about 0.5 the frequency of maximum scattering is the average plasma frequency divided by $\sqrt{2}$. For values of $\alpha$ greater than 0.5 the frequency of maximum scattering is very nearly equal to the plasma frequency at the edge of the plasma column.

4–10 Guided wave propagation along a cylindrical plasma column. The third experiment, shown schematically in Fig. 4–22, deals with the space-charge wave propagation discussed in Section 4–3. Here, however, the electrons have only a very small drift velocity, since they are the electrons of a mercury arc discharge plasma. The qualitative behavior of these waves was described in connection with Fig. 4–4, where the dependence of the oscillation frequency upon the wavelength of the disturbance was discussed. This line of reasoning leads to a dispersion curve ($\omega$ versus $\beta$) of the type shown in Fig. 4–23, for no magnetic field [81]. The lower branch has a nonzero group velocity, indicating that signals can be propagated along the plasma column. The upper branch, which represents the internal plasma oscillations in which there is no electric field outside the column, has zero group velocity and the disturbances remain localized. The application of a longitudinal magnetic field modifies the results slightly.

In the experiment (Fig. 4–22) a discharge is sustained between the cathode and the collector and the radiofrequency signal is launched on

Fig. 4-22. Schematic drawing of apparatus for measuring wavelength and attenuation in a plasma-filled waveguide.

Fig. 4-23. $\omega-\beta$ diagram of lowest space-charge mode (heavy line) of propagation and plasma resonance (light line) of an ionized column in free space.

The ionized column by the collector itself. The discharge tube is surrounded by a slotted metallic cylinder. The amplitude and phase of the propagating wave are sampled by a small electric probe which can be moved along the tube in the slot. By comparing the signal picked up by this probe with a reference signal from the oscillator driving the input, the phase velocity of the space-charge wave can be measured. Typical experimental results are compared with a theoretical curve in Fig. 4-24. The
vertical coordinate is frequency, and it has been normalized by dividing by the cyclotron frequency $\omega_c$. The horizontal coordinate is the propagation constant $\beta$, normalized by multiplying it by the radius $a$ of the column. The cyclotron frequency was 250 mc, corresponding to a magnetic field of about 100 gauss, and the electron plasma frequency was approximately 500 mc. Propagation in both bands is clearly discernible. The effect of the magnetic field on the upper band is to couple the oscillations in various parts of the plasma and make the frequency of oscillation depend slightly on wavelength. The electric field outside the plasma
column is still very weak, and measurements of a characteristic of this band are difficult. Note particularly that it is a backward wave, that is, phase and group velocities are in opposite directions.

In the analysis of this experiment several simplifying assumptions have been made:

(1) The electron temperature is assumed to be zero. So long as wave velocities are much greater than mean thermal velocities, this should be a good approximation.

(2) Electron collisions have been neglected. This is probably the poorest of all the approximations, since the electron collision frequency is of the order of 100 me.

(3) The ion motion was neglected. At very low radiofrequencies ion motions may become quite important, and it would be of interest to trace the behavior of this wave through the transition to a hydromagnetic wave.

(4) It is assumed that the electric field can be derived from a scalar potential. This is equivalent to the neglect of retardation effects and should be a satisfactory approximation so long as the velocity of the waves is much less than the velocity of light. In the experiment this condition is fulfilled.

Thus the analysis does not predict the existence of the waveguide modes, for the geometry used in the experiment at the cutoff frequency of the lowest waveguide mode is approximately 15,000 mc. Propagation of the space-charge waves, on the other hand, has been studied in the frequency range 100 to 2000 mc. The analysis of these waves is facilitated by representing the properties of the plasma by a dielectric tensor:

\[
\epsilon = \epsilon_0 \begin{pmatrix}
1 + \frac{\omega_p^2}{\omega_c^2 - \omega^2} & \frac{j \omega \omega_p^2}{\omega (\omega_c^2 - \omega^2)} & 0 \\
\frac{-j \omega \omega_p^2}{\omega (\omega_c^2 - \omega^2)} & 1 + \frac{\omega_p^2}{\omega_c^2 - \omega^2} & 0 \\
0 & 0 & 1 - \frac{\omega_p^2}{\omega^2}
\end{pmatrix}
\]  

(4-35)

Since the plasma frequency appears as a parameter in ω-β characteristics, phase velocity measurements provide a method of determining electron density. For example, in Fig. 4-24 the experimental points are in close agreement with the theoretical curve \( \omega_p/\omega_c = 1.9 \), and hence the electron plasma frequency is 1.9 × 250 mc. The points marked W in Fig. 4-21 were obtained in this way. A particular advantage of this method is that measurements can be made at frequencies at least an order
of magnitude less than \( \omega_p \), and this may be of considerable use in studying high density plasma. The phase characteristic of Fig. 4-23 for no static magnetic field has the asymptotic low-frequency characteristic

\[
\omega = \beta \cdot \sqrt{\frac{\log b/a}{2K_e}} \omega_p, \quad \omega \ll \omega_p, \quad \omega / \beta \ll c, \quad (4-36)
\]

where \( a \) is the plasma radius, \( b \) the radius of the surrounding metallic cylinder, and \( K_e \) is the relative dielectric constant of the medium between \( a \) and \( b \). An analysis of a nonuniform plasma in which the density variation is given by Eq. (4-34) shows that the low-frequency asymptotic phase velocity is proportional to the average plasma frequency \([91]\). Hence when (4-36) is applied to calculate electron density, it yields average density.