RESONANT BEHAVIOUR OF
A PLASMA SLAB-CONDENSER SYSTEM

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Résumé

Un modèle particulier de plasma, tranche de plasma à densité d’équilibre constante placée dans un condensateur, est étudié. Lorsque la température est nulle, on voit apparaître une fréquence de résonance géométrique caractéristique \( \omega_r \sqrt{l-a}/l \) dépendant des dimensions relatives du système physique. Lorsque la température n’est pas nulle, en plus de la fréquence fondamentale, légèrement modifiée par la température, apparaît un spectre de résonance discontinu caractérisé par

\[
\omega_r^2 \simeq \omega_e^2 \left[ 1 + \frac{(2N + 1)^2 \gamma^2 \epsilon_0^2}{a^2} \right].
\]

\( N = 1, 2, 3, \ldots \). Ce dernier cas est ensuite comparé à l’étude d’un plasma cylindrique étudié par R. W. Gould.

1. Introduction. Considerable interest has arisen since 1951 concerning the resonant behaviour of a cylinder of plasma\(^1\)\(^2\)\(^3\)\(^4\)\(^5\). Much theoretical work\(^6\)\(^7\)\(^8\)\(^9\)\(^10\) has been done on this subject but only the first resonance given by \( \omega_e \simeq \omega_e / \sqrt{l} \) is perfectly understood. Here \( \omega_e \) denotes the plasma electron angular frequency given by \( \omega_e = \langle N \rangle e^2 \epsilon / \epsilon_0 m \) where \( -e, m \) and \( \langle N \rangle \) are the charge, mass and equilibrium density of electrons and \( \epsilon_0 \) is the permittivity of vacuum. The object of the present paper is to study a plasma slab placed in an infinite plane condenser and particularly its resonant behaviour. The problem is characterized by the constant density \( \langle N \rangle \) of electrons and by two lengths \( l \) and \( a \). Two cases are considered:

(a) \( T = 0 \) (\( T \): Kelvin temperature) and a characteristic geometrical frequency \( \omega_r \sqrt{l-a}/l \) appears;

(b) \( T \neq 0 \), the fundamental characteristic frequency \( \omega_r \sqrt{l-a}/l \) is slightly altered and a discrete resonance spectrum

\[
\omega_n^2 \simeq \omega_e^2 \left[ 1 + \frac{(2N + 1)^2 \gamma^2 \epsilon_0^2}{r_D^2} \right].
\]

\( N = 1, 2, 3, \ldots; r_D \): Debye radius) is obtained.

— 357 —
This last case is then compared with the study made by R. W. Gould\(^9\) of the cylindrical plasma with constant \(\langle N \rangle\) and \(T \neq 0\). One may then conclude that a study is needed where both the assumptions \(\langle N \rangle \neq \) constant and \(T \neq 0\) are made\(^10\)\(^10\).

![Fig. 1. The plasma slab-condenser system.](image)

2. **Linearized equations of the plasma slab.** The plasma is placed in a container of width \(a\) and both other dimensions are infinite. The container is placed between two infinite condenser plates \(P_1\) and \(P_2\) and there exists a difference of potential \(V(t)\) between \(P_1\) and \(P_2\). \(E(x)\) is the electric field in the plasma and \(E_{\text{ext}}(x)\) is the field outside the plasma but between \(P_1\) and \(P_2\). The distance \(P_1P_2\) is equal to \(l\).

The ions are assumed fixed and the linearized hydrodynamic equation of motion\(^11\) for the electrons is:

\[
m\langle N \rangle \frac{\partial u}{\partial t} = -e\langle N \rangle E - n\gamma kT \frac{\partial n}{\partial x} - n\langle N \rangle^2 \dot{u} \tag{1}
\]

where \(m\) and \(-e\) are the mass and charge of the electron, \(N = \langle N \rangle + n\) the density, \(\gamma\) is a coefficient depending on the assumptions made for the perturbation law of the electronic gas, \(n\langle N \rangle^2 \dot{u}\) is the collision term and \(\dot{u}\) the mean macroscopic velocity in the \(x\) direction (in the two other directions this velocity is zero because of symmetry).

The Maxwell equations give:

\[
\begin{align*}
(i) & \quad -\langle N \rangle e u + e_0 \frac{\partial E}{\partial t} = J(t) = e_0 \frac{\partial E_{\text{ext}}}{\partial t} \\
(ii) & \quad \int_0^a E\, dx + \int_a^l E_{\text{ext}}\, dx + V(t) = 0
\end{align*}
\tag{2}
\]

because \(\text{div curl } \overrightarrow{H} = \text{div} [j + e_0 (\partial E/\partial t)] = 0\) and the total current is noted \(J\).

We notice immediately that \(E_{\text{ext}}\) is independent of \(x\)

\[
\int_0^a E\, dx + \int_a^l E_{\text{ext}}\, dx + V(t) = 0 \tag{3}
\]
because the static approximation is made

\[ \frac{\partial n}{\partial x} = -\frac{\partial u}{\partial x} \]  

(4)

It may be sometimes more convenient to use the combination of (2) and (4), i.e. the linearized continuity equation

\[ \frac{\partial n}{\partial t} + \langle N \rangle \frac{\partial u}{\partial x} = 0 \]  

(5)

3. Cold plasma \((T = 0)\). We will neglect the term in \(\partial n/\partial x\), that is we will take a plasma with \(T = 0\). We perform a Laplace transform, characterized by the variable \(\rho\), on the time. The Laplace transforms of the functions considered are distinguished by an index \(\rho\) and the initial values for \(t = 0\) by an index zero.

\[ m\langle N \rangle [\rho u_\rho - u_0] = -\langle N \rangle E_\rho - \alpha\langle N \rangle^2 u_\rho \]  

(6)

\[ -\langle N \rangle \rho u_\rho + \rho_0 [\rho E_\rho - E_0] = \rho_0 [\rho E_{\text{ext} \rho} - E_{\text{ext} 0}] = J_\rho \]  

(7)

\[ \int_0^a E_\rho \, dx + E_{\text{ext} \rho}(l-a) + V_\rho = 0 \]  

(8)

\(u_\rho\) is introduced from (6) into (7) to obtain \(E_\rho\) and \(E_{\text{ext} \rho}\) which are then introduced into (8). This substitution gives:

\[ J_\rho = -\frac{\varepsilon_0}{\rho} \left( \frac{V_\rho + E_{\text{ext} 0}}{l-a} + \langle N \rangle \frac{u_\rho}{\varepsilon_0} \int_0^a \frac{\rho_0}{\omega_c^2 + \rho^2 + \rho x' + \rho \rho'} \, dx + (\rho + \rho') \int_0^a \frac{E_\rho}{\omega_c^2 + \rho^2 + \rho x'} \, dx \right) \]  

(9)

\[ \rho' = \rho \omega_c^2 \quad \text{and} \quad \omega_c^2 = \frac{\langle N \rangle \varepsilon_0^2}{\varepsilon_0 \alpha m} \]

We will now examine a few interesting cases when \(E_{\text{ext} 0} = u_0 = E_0 = 0\).

We will neglect collisions \((\varepsilon' = 0)\).

(a) \(V(t) = A \delta(t)\).

\[ J_\rho = -\frac{\varepsilon_0 A}{l} \frac{\rho (\omega_c^2 + \rho^2)}{\rho^2 + (\frac{l-a}{l}) \omega_c^2} \]  

because \(V_\rho = A\).

Thus:

\[ J(t) = -\frac{\varepsilon_0 A}{l^2} \omega_c^2 \cos \omega_c \theta \]  

with \(\theta = \sqrt{\frac{l-a}{l}}\) and \(0 < \theta < 1\)  

(10)
Let us call \( \omega_0 \) the characteristic geometrical frequency of the plasma. The adjective geometrical indicates that this frequency depends on the relative dimensions of the physical system.

**Remark 1:** If \( \ell = a \), \( J(t) = -\frac{\varepsilon_0 A}{l} [\omega_e^2 U(t) + \delta'(t)] \) where \( U(t) \) is the unit function and \( \delta'(t) \) the derivative of the delta function. In this case, the system condenser-plasma behaves like a parallel LC circuit and the plasma has an equivalent coefficient of self inductance which is proportional to \( \omega_e^{-2} \).

(b) \( V(t) = A e^{-i\omega_0 t} \) for \( t > 0 \).

\[
J_p = -\frac{\varepsilon_0 A}{l} \frac{\dot{\phi}(\omega_e^2 + \theta^2)}{(\theta^2 + \theta^2 \omega_e^2)(\dot{\phi} + i\omega)} \quad \text{because} \quad V_p = \frac{1}{\dot{\phi} + i\omega}
\]

After some algebra, one finds:

\[
J(t) = -\frac{i\varepsilon_0 A}{l(\theta^2 + \omega^2)} [\omega_e^2 - \omega^2 - \omega_e^2(1 - \theta^2)(i\theta \omega_e \sin \omega_e \theta - \omega e \cos \omega_e \theta)] \quad \text{(11)}
\]

Both the applied frequency and the characteristic geometrical frequency appear in the response. If \( \omega = \omega_e \), the geometrical frequency appears alone.

(c) \( V(t) = A e^{-i\omega_0 t} \) for \( t > 0 \).

Formula (11) gives then \( \frac{0}{0} \) and the real value is:

\[
J(t) = -\frac{i\varepsilon_0 A}{2\theta l} [-(i\omega_e)(1 - \theta^2) \sin \omega_e \theta - 2\theta \omega e \cos \omega_e \theta - (i\omega_e^2 - \theta^2) e^{-i\omega_0 t}] \quad \text{(12)}
\]

There is a resonance phenomenon and \( J(t) \) diverges linearly with \( t \).

**Remark 2:** If \( E_{ext0}, E_0 \) and \( u_0 \neq 0 \), the poles of \( J_p \) are unchanged and the frequencies appearing in \( J(t) \) remain thus the same. This is only true of course if \( E_{ext0}, E_0 \) and \( u_0 \) have no poles of their own.

**Remark 3:** If the collisions are taken into account (\( \alpha' \neq 0 \)), the poles are given by \( \dot{\phi}^2 + \alpha' \dot{\phi} + \theta^2 \omega_e^2 = 0 \) or

\[
\dot{\phi} = -\frac{\alpha'}{2} \pm i\omega_e \sqrt{1 - \frac{\alpha'^2}{4\theta^2 \omega_e^2}} \quad \text{or} \quad -\frac{\alpha'}{2} \pm i\omega_e \left(1 - \frac{\alpha'^2}{4\theta^2 \omega_e^2}\right)
\]

The characteristics of the response are the same except for the appearance of a damping term \( e^{-\alpha'/2\ell} \) and a slight correction to the geometric frequency. Note that this correction is the same as the one obtained for an infinite plasma (see 11), p. 150).

4. **Warm Plasma** \((T \neq 0)\). We will not perform a Laplace transform for \( T \neq 0 \) because this would bring unnecessary difficulties with the initial conditions now that the term with \( \dot{\ell}/\ell \) appears. We will limit ourselves
to the case $V(t) = V e^{-i\omega t}$. Noting $E = -\partial \phi / \partial x$, equations (1), (2) and (4) give the following formula when collisions are neglected:

$$
\left( \frac{\partial^2}{\partial x^2} + k^2 \right) \frac{\partial^2 \phi}{\partial x^2} = 0
$$

with

$$
k^2 = \frac{\omega^2 - \omega^2_e}{w^2} \quad \text{and} \quad w^2 = \frac{\gamma KT}{m}
$$

(13 bis)

The general solution of (13) is:

$$
\phi = Ax + B + C \cos kx + D \sin kx
$$

(14)

Introducing (14) into (3):

$$
A \sigma = C (\cos kx - 1) + D \sin kx - E_{\text{ext}}(\beta - a) - V
$$

(15)

The boundary condition $u = 0$ for $x = 0$ and $x = a$ will now be applied to (1) and (2). (2) is written:

$$
-i\omega_e E_{\text{ext}} = -i\omega_e (-A + C k \sin kx - D k \cos kx) = J = -i\omega_e E_{\text{ext}}
$$

(i) $u = 0$ for $x = a$ gives:

$$
-i\omega_e E_{\text{ext}} = -i\omega_e (-A + C k \sin kx - D k \cos kx) = I
$$

(16)

(ii) $u = 0$ for $x = 0$:

$$
E_{\text{ext}} = -A - D k
$$

(17)

(16)–(17) gives:

$$
D (\cos kx - 1) = C \sin kx
$$

(18)

The cases $ka = 2N\pi$ or $(2N + 1)\pi (N = 0, 1, 2, \ldots)$ will be examined later.

After introducing

$$
\frac{\partial^2 E}{\partial x^2} = -\frac{e}{\varepsilon_0} \frac{\partial n}{\partial x}
$$

(1) takes the following form:

$$
i o m \langle N \rangle u = e \langle N \rangle [-A + C k \sin kx - D k \cos kx] + \\
+ \frac{\varepsilon_0 \gamma KT}{e} [C k^3 \sin kx - D k^3 \cos kx]
$$

(19)

(iii) $u = 0$ for $x = 0$:

$$
D = \frac{e^2 \langle N \rangle E_{\text{ext}}}{k^3 \varepsilon_0 \gamma KT}
$$

(20)

(iv) $u = 0$ for $x = a$ gives an identity as one expects.
Let us introduce $A$, $C$ and $D$ as functions of $E_{\text{ext}}$, i.e. $f$, in (15) in order to find a relation between $f$ and $V$. One obtains:

$$\frac{i f}{\varepsilon_0 \gamma} \left[ -\frac{\ln \varepsilon_0 \gamma KT}{k^2 \varepsilon_0 \gamma KT} \left( 2 \tan \frac{k a}{2} - k a \right) \right] = V \tag{21}$$

The “impedance” $Z$ (in fact impedance multiplied by the unit of surface of the condenser plates) of the condenser-plasma system is given by:

$$Z = \frac{i m \omega^2}{\varepsilon_0 \gamma} \left[ -\frac{\ln \varepsilon_0 \gamma KT}{m \omega^2} + 2 \tan \frac{k a}{2} - k a \right] \tag{22}$$

The resonance frequencies are given by the zeros of $Z$. Putting $x = k a$ and $k T / m \omega^2 = r_D^2$ where $r_D$ is the Debye radius, $Z = 0$ is given by the solutions of:

$$\frac{x}{2} + \frac{4 \gamma r_D^2}{a^3} \left( \frac{x}{2} \right)^3 = \tan \frac{x}{2} \tag{23}$$

For $\omega < \omega_0$, $k^2 < 0$; we put $k' = i k$, $x' = i x$ and (23) becomes:

$$\frac{x'}{2} - \frac{4 \gamma r_D^2}{a^3} \left( \frac{x'}{2} \right)^3 = \tan \frac{x'}{2} \tag{24}$$

Because the $x'/2$ is an increasing function of $x'/2$, because

$$\frac{x'}{2} - \frac{4 \gamma r_D^2}{a^3} \left( \frac{x'}{2} \right)^3$$

goes through a maximum and then decreases and that at the origin the $x'/2$ is above

$$\frac{x'}{2} - \frac{4 \gamma r_D^2}{a^3} \left( \frac{x'}{2} \right)^2 \quad \text{as} \quad \frac{r_D}{a} \ll 1,$$

there exists a single solution to (24) for $x'$ real and positive (see fig. 2).

![Fig. 2. Graphical solution of equation (24).](image-url)
A typical value of \((r_0/a)^2\) is 1/1600 and taking \(4r_l/a = 16\), one finds that the solution of (24) is given by \(x'/2 = \theta\). If, instead of the intersection of
\[
\frac{x'}{2} - \frac{4r_l r_p^2}{a^3} \left( \frac{x'}{2} \right)^3
\]
with the \(x'/2\), the intersection with the axis is sought, one finds \(x'/2 = 10\). This last remark enables to write the root of (24) with an excellent approximation:
\[
\frac{x'}{2} - \frac{4r_l r_p^2}{a^3} \left( \frac{x'}{2} \right)^3 = 0. \tag{25}
\]
Thus, using (13 bis), the first resonance frequency of the plasma is given by:
\[
\omega_0 \cong \omega_c \sqrt{\frac{l-a}{l}} = \omega_c \theta. \tag{26}
\]

![Graphical solution of equation (23).](image)

When \(\omega > \omega_c\), we look graphically for the solutions of (23). Fig. 3 shows that the solutions are approximately given by
\[
x_N \cong (2N + 1) \pi \quad N = 1, 2, 3, \tag{27}
\]
or
\[
k_N \cong (2N + 1) \frac{\pi}{a} \quad N = 1, 2, 3, \ldots
\]
The approximation of (27) becomes better and better as \(N\) increases. For
\[
N = 1, \quad k_1 = \frac{3\pi - 0.19}{a}
\]
instead of \(3\pi/a\) and the approximation is already quite good.
The resonance frequencies are then given by
\[ \omega_{N}^{2} \equiv \omega_{e}^{2} \left[ 1 + \frac{(2N + 1)^{2} \pi^{2} \gamma r_{D}^{2}}{\alpha^{2}} \right] ; \quad N = 1, 2, 3, \ldots \quad (28) \]

We must now study the cases when \( ka = 2N\pi \) and \( ka = (2N + 1)\pi \).

(i) \( ka = 2N\pi ; \quad N = 0, 1, 2, \ldots \)

(15) gives
\[ Aa - E_{\text{ext}}(l - a) = V \quad (29) \]

Taking into account (17), (20), (29) and \( J = -i\omega_{e}E_{\text{ext}} \), one obtains:
\[ \frac{a}{i\omega_{e}o} \left[ \frac{e^{2}N}{k^{2}e_{0}yKT} + \frac{l}{a} \right] J = V \quad (30) \]

\( Z = 0 \) leads then to
\[ k^{2} - \frac{a}{l} e^{2}N \]

and this cannot be true for real values of \( k \). Thus, no further resonance frequencies appear for \( ka = 2N\pi \).

(ii) \( ka = (2N + 1)\pi ; \quad N = 0, 1, 2, \ldots \)

(16) and (17) give:
\[ D = 0 \quad (31) \]
\[ E_{\text{ext}} = -A \quad (32) \]

(19) gives for \( x = 0 \) (\( u = 0 \))
\[ A = 0 \quad (33) \]

Then (13) gives
\[ C = -\frac{V}{2} \quad (34) \]

Thus
\[ E = -\frac{V}{2}k \sin kx \quad (35) \]

As \( E_{\text{ext}} = J = 0, \ Z = \infty \) and we observe an anti-resonance.

Furthermore the resonances given by the approximate formulae (27) are extremely sharp as the domain of \( k \) between resonance and anti-resonance is very small.

\textbf{Remark 1.} Let us examine the case when \( T \to 0 \) (28) shows that \( \omega_{N+1} - \omega_{N} \to 0 \) because \( r_{D}^{2} = KT/m\omega_{e}^{2} \), that is to say that the resonance frequencies gather towards \( \omega_{e} \). (21) shows that \( \omega_{e} \) is not a resonance frequency.
The isolated resonance frequency will be obtained when \( k^2 < 0 \). Putting 
\( k = i k', \ k' \to \infty \) when \( T \to 0 \) and through (21) we see that \( \omega \to \omega_0 \theta \) because \( Z = 0 \) gives

\[
 l = \left( 2 \text{ th} \frac{k'a}{2} - k'a \right) \frac{\omega_0^2 a}{\omega^2 - \omega_0^2} \tag{36}
\]

Remark 2. The present model can be investigated experimentally. It is possible to make experiments with the following set-up for the condenser plates \( P_1 \) and \( P_2 \) (see fig. 4.):

![Diagram](image1)

**Fig. 4.** Condenser plate consisting of parts A and B with current \( JS_d \) being read.

Each plate is divided into two parts \( A \) and \( B \) isolated from one another. \( A \) and \( B \) are both brought to the potential \( V \) but only the current \( I = JS_A \) (where \( S_A \) is the surface of \( A \)) contributed by part \( A \) is read.

The results of \( I \) relative to \( A \) will be very near to those of a plasma infinite in the directions \( y \) and \( z \) as the boundary effects are practically eliminated. An experimental difficulty is the realization of a rectangular discharge tube and of the U.H.F. condenser and detecting system. This problem is now being studied at the Ecole Royale Militaire as it has been reported earlier\(^{14}\).

Remark 3. When the present study was completed, we saw a paper of P. A. Wolff\(^{12}\) on the theory of plasma resonances in solids. This work was in fact the \( T \neq 0 \) continuation of a work\(^{13}\) of Dresselhaus, Kip and Kittel on the same subject. Whilst Dresselhaus, Kip and Kittel found a single resonance at \( \omega_n \), Wolff looked at the influence of a non zero temperature. He considered a solid indefinite in two directions of thickness \( a = 2d \) in a radio frequency field \( E_0 \) normal to the surface. He found resonances for

\[
 kn = (2n + 1) \pi \quad n = 0, 1, 2 \tag{37}
\]

which correspond to the following resonance frequencies:

\[
 \omega_n^2 = \omega_0^2 + \frac{\langle \alpha^0 \rangle_{av}}{3} \frac{(2n + 1)^2 \pi^2}{a^2} \tag{38}
\]

Wolff’s work is thus a special case of ours and corresponds in fact to our
anti-resonances. Wolff's case has of course no characteristic geometrical frequency.

5. **Comparison with the cylindrical plasma** ($\langle N \rangle = \text{constant and } T \neq 0$).

*Conclusions.* When $T = 0$, it is well known that a cylindrical plasma with constant equilibrium density has a single resonance frequency equal to $\omega_0/\sqrt{2}$.

R. W. Gould\(^9\) studied the case $T \neq 0$. The equation that had to be solved was again

$$ (V^2 + k^2) V^2 \phi_l = 0 $$

(39)

where $\phi_l$ is the potential inside the plasma. Outside the plasma, the potential obeys of course $V^2 \phi_0 = 0$. The boundary conditions chosen for $r = a (a$: radius of the cylinder of plasma) are:

1. $\phi$ continuous
2. $\frac{\partial \phi}{\partial r}$ continuous
3. zero radial velocity

The following dispersion equation is then obtained:

$$ 1 + \frac{2\omega^2}{\omega_e^2} + \frac{J_1(ka)}{ka J_1(ka)} = 0 $$

(40)

where $J_1$ is the Bessel function of the first kind of order 1. This gives the following resonance frequencies:

$$ \omega_0 \sim \frac{\omega_e}{\sqrt{2}} $$

(41)

$$ \omega^2 - \omega_0^2 \left[ 1 + \gamma \left( \frac{r_P}{a} \right)^2 x_n^2 \right] \quad \text{with} \quad n = 1, 2, 3, \ldots $$

(42)

and $x_n = 5.1; 8.4; 11.6; \ldots$

Going back to (26) and (28) it is seen that the one-dimensional model gives, as expected, very comparable results to those obtained for the cylindrical plasma in the same approximation. The intervals between the resonant frequencies are of the same order of magnitude.

Now, the experimental results\(^1\)\(^2\)\(^3\)\(^4\)\(^5\) in the cylindrical case show one main resonance and a set of secondary ones. The frequency intervals observed experimentally are however several times greater than those obtained theoretically by Gould. The study of the plasma slab-condenser system thus confirms the idea that an agreement between theory and experiment in the case of the cylindrical plasma might be obtained when $T \neq 0$ and when one or more of the assumptions forming the basis of the present paper will be altered. As one of the assumptions, i.e. the constant
equilibrium density, is certainly not verified in the experimental set-ups, it appears extremely interesting to study \( \langle N \rangle \neq \text{constant} \). This will only be possible after a critical study of the equations that are to be used once a consistent model is adopted\(^{10}\)\(^{15}\)\(^{16}\).

Beyond the comparison with the cylindrical case we will retain from the present paper the appearance of a characteristic geometrical frequency

\[ \omega_c \sqrt{\frac{l-a}{l}} \]

which can vary at will between 0 and \( \omega_c \).

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