LARGE SIGNAL THEORY OF BEAM-PLASMA AMPLIFIERS

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Introduction.

In order to predict and evaluate the operation of experimental beam-plasma tubes at high power levels and saturated power outputs it is necessary to carry out a large-signal analysis. The method described here is based on the methods of Tien, et al 1 and Webber 2 in which the electron beam is assumed to be divided into a finite number of discs with a uniform distribution of charge on the disc. These discs are followed along their trajectories -- a step along each trajectory is calculated and after each time interval, the locations of all the discs in a given period are found and the field at each disc due to every other disc is calculated. This enables the calculation of trajectories to be carried one step further. At any time, the RF current content and velocity of the discs may be computed. In order that this analysis be applied to the beam-plasma system, it is necessary to construct a suitable force law between discs in a plasma. In this paper, we will describe how this has been accomplished. This force law enables the large signal analysis to be carried out. The analysis is programmed for computer solution and some results of this analysis are presented with particular reference to efficiency considerations and harmonic content of the electron beam interacting with the plasma. Results are also compared with experiment.

Assumptions.

The situation which is analysed is shown schematically in Figure 1. The electron beam passes in turn through an input cavity resonator, a plasma-free drift region, the plasma interaction region, another plasma-free drift region, and finally an output cavity resonator to the collector. In order to make the non-linear analysis possible the following important simplifying assumptions are made:

I. The plasma remains linear and only the motion of the beam electrons is treated in an exact non-linear manner. That this is a good assumption may be seen from an examination of the equations of motion for the electrons both in the beam and in the plasma where it is found that the second order terms in the equations of motion become significant in the beam equations when they are still negligible in the plasma equations of motion. This can be understood physically since the same fields are supported in both the plasma and the beam, but the space-charge restoring forces are greater in the plasma thereby inhibiting the plasma particles from making as large RF excursions as the beam particles.

II. The plasma is spatially uniform with sharp boundaries separating if from the drift regions and the possible reflections of plasma waves from these boundaries are neglected. The plasma is assumed to be isotropic, that is zero magnetic field is assumed for the plasma. This is reasonable since the cyclotron frequency of the applied magnetic field to focus the electron beam is very much less than the plasma frequency of the electrons in the plasma under study. It will be shown later how the case of the axially non-uniform plasma which occurs in practice under normal experimental conditions is treated.

III. The electron beam is approximated by a set of rigid discs of charge which can have free axial motion. Although the electric field may vary across the disc and there may be transverse components of the electric field, no transverse motion is permitted and all parts of the disc move with the same axial velocity. This is equivalent to the assumption of an infinite magnetic field for the electron beam. This is compatible with the previous assumption since the cyclotron frequency of the applied magnetic field is always very much greater than the plasma frequency of the beam electrons.

IV. In the drift region each disc exerts a force on every other disc which decreases exponentially with the separation between discs. Both assumptions III and IV have been employed successfully in large-signal, traveling-wave tube theory by Tien, et al 1.

Method of Analysis.

The motion of \( N \) particles per cycle are followed, the position and velocity of the \( n \)-th particle being denoted by \( z_n \) and \( v_n \), respectively. Each disc is uniformly charged and has a radius \( b \), charge \( q \) and a mass \( m_0 \). The equations of motion of the \( n \)-th particle are

\[
v_n = \sum_{m \neq n} a_{nm} \quad (1)
\]

\[
\dot{z}_n = v_n \quad (2)
\]

where \( a_{nm} = 1/m_0 \) times the force exerted on particle \( n \) by particle \( m \). In the drift region, an approximation to this force law between the two discs is given by

\[
\alpha_{nm} = \frac{1}{2 \pi \varepsilon_0 m_0 b^2} \frac{q^2}{v_n - v_m} \exp(-|x|/b) \quad (3)
\]

where \( \alpha \) is related to the plasma reduction factor \( R \) by

\[
R^2 = \frac{1}{1 + \alpha^2} \left( \frac{\omega}{\beta_\omega} b \right) \quad (4)
\]

The quantity \( \beta_\omega \) is the electronic propagation constant, \( \omega/v_0 \), where \( \omega \) is the applied radian frequency and \( v_0 \) is
the beam velocity. It is necessary to evaluate this force $a_{em}$ in the presence of a plasma.

**Force Law Between discs in a Plasma.**

The force law between electron discs in a plasma will be quite different from that which pertains to a vacuum. This force law is calculated for a single particle traversing a plasma at a constant velocity. The method used follows the theory of Bohm and Pines 3 which makes extensive use of space-time Fourier transforms. The use of transforms is possible since we have assumed the plasma remains linear and super-position is valid.

The dielectric constant for a warm plasma which includes the electron thermal speeds and the dissipative effects of short range collisions is given by 3

$$K(\omega, k) = 1 + \chi(\omega, k) = 1 - \frac{\omega_p^2}{\omega(\omega - iv) - k^2 w^2}$$

(5)

where $\omega_p, v$ and $w$ are the electron plasma frequency, collision frequency and electron thermal speed respectively and $\chi$ is a generalized susceptibility. The quantities $\omega$ and $k$ are the Fourier transform variables, frequency and vector wave number respectively. Poisson's equation for the potential is solved by Fourier transform methods by writing

$$\varphi(r, t) = \int \tilde{\varphi}(k, \omega) e^{i(\omega t - k \cdot r)} \frac{d^3 k}{(2\pi)^3} d\omega$$

(6)

where

$$\tilde{\varphi}(k, \omega) = \int \varphi(r, t) e^{-i(\omega t - k \cdot r)} d^3 r dt$$

(7)

is the Fourier transform of the potential. The Fourier transform of Poisson's equation is

$$-k^2 \tilde{\varphi} = - (\tilde{\rho}_{\text{plasma}} + \tilde{\rho}_{\text{source}})/\varepsilon_0$$

(8)

where $\tilde{\rho}_{\text{source}}$ and $\tilde{\rho}_{\text{plasma}}$ are the Fourier transforms of the charge density of the moving charges and the induced charge density in the plasma respectively. We have

$$\tilde{\rho}_{\text{plasma}} = + k^2 \varepsilon_0 \varphi$$

(9)

and the Fourier transform of the source charge density of a moving point charge may be shown to be given by

$$\tilde{\rho}_{\text{source}} = 2 \pi q \delta(\omega - k \cdot v).$$

(10)

The equation for the Fourier transform of the potential is

$$\tilde{\varphi} = \frac{2 \pi q \delta(\omega - k \cdot v)}{k^2 \varepsilon_0 K(\omega, k)}.$$

(11)

From this potential may be determined the inter-particle force law for various values of the frequency by transforming back and numerically evaluating the integrals with proper regard to the various singularities which arise.

A test particle moving through the plasma experiences a force due to the disturbance fields set up in the plasma by the passage of other particles, and in particular this disturbance is in the form of plasma oscillations which are damped by thermal and collisional effects. The cumulative effect of all the beam particles is to induce a disturbance in the plasma at the driving frequency and, if this driving frequency is close to the plasma frequency, the induced disturbance is very large and completely alters the space-charge forces in the beam. This can be seen by examining Figure 2 which is the inter-particle force law computed for given beam and plasma conditions for various frequencies. This case is for a zero temperature plasma with the damping of disturbances being entirely a collisional effect. A series of discs are sent into the plasma every $2 \pi/\omega$ seconds and a test disc moving with the beam velocity is located somewhere between a pair of discs as is illustrated by Figure 2 a. The force is shown on this test disc due to a given disc and all discs similar to this given disc but which pass by one, two, three, etc., cycles earlier. This force is related to the disturbance fields which are set up. The first curve which is shown in Figure 2 b is essentially the non-plasma situation since the plasma frequency is very much less than the operating frequency. At $z = 0$, the test disc is just behind one of the series of discs and is repelled away to the left and at $z$ maximum it would be just ahead of the next one in the series and would still be repelled away, but this time to the right, and hence the change in sign of the force. The force vanishes half-way between the two discs since the test disc would see equal and opposite forces from each disc. For Figure 2 c the frequency is now below the plasma frequency and it is seen that although the test disc is repelled at very short range, it experiences an attractive force at larger ranges. In other words, space-charge repelling forces on the discs have changed sign due to the disturbance fields in the plasma and now the discs tend to fall together into bunches. The curve of Figure 2 d shows the effect at a frequency much below the plasma frequency where there are still regions of attractive forces but not over as large a region as for $\omega/\omega_p = 0.9$. In the computer program, for given beam and plasma parameters, the appropriate force law is calculated and stored in tabular form as a function of position and this is used for solving the equations of motion in the plasma when the force on a disc due to every other disc is computed.

**Results.**

In practice, in our experimental tubes 4 the plasma is non-uniform axially with the number density being closely approximated by a sine function. In our large
signal analysis this situation is approximated by constructing the plasma of several regions in which the plasma frequency can be varied from region to region. The computations were carried out with an IBM 7094 Model II computer and a Stromberg-Carlson 4020 Plotter.

Figure 3 shows a typical case of an axially non-uniform plasma studied. At the input of the tube, velocity modulation is applied, and the beam drifts for 15 electronic wavelengths. The plasma region consists of five steps of four electronic wavelengths each. Following the end of the plasma there is a drift region. The beam parameters correspond to the beam in the experimental tube where the beam perveance is $1 \times 10^{-6}$, voltage is 20 kV, and diameter is 0.5 inches. Figure 4 shows the electron trajectories with 32 discs in a frame of reference moving with the average drift velocity of the beam. It is noted that, in the plasma, the particles are accelerated toward the bunch rather than away from the bunch as is the case of space-charge debunching and very tight bunches are obtained. Although the plasma is terminated at $\beta_z z = 35$ the electrons still move together towards the bunch and give the best bunching positions around $\beta_z z = 40$. The harmonic content of the beam current is shown in Figure 5. With the fundamental current, second and third harmonics growing to large values, the theoretical maximum of perfect or delta function bunching of twice the d.c., current in the harmonics is approached. The velocity, time coordinates of the 32 discs

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**Fig. 3.** — Configuration of the Non-uniform Plasma.

**Fig. 4.** — Disc trajectories for configuration of Figure 3.
for various electronic wavelengths along the beam are shown in Figure 6 for a series of values of $\beta_z z$. At the best bunching point around $\beta_z z = 40$ the velocity spread is small. This small velocity spread occurs since the beam left the plasma before the best bunching was achieved. If the electrons are allowed to continue to their best bunching point within the plasma, it is found that the velocity spread is increased with consequent lower efficiencies.

In the experimental tube $^4$, saturated gain represented by this configuration was about 40 db which is close to the value predicted. The computed velocity spread predicts efficiencies of over 40% which is in approximate agreement with experimental observations $^5$. Also, for a fixed input level when the frequency is varied over a 20% bandwidth the current modulation at $\beta_z z = 42$ remains within 30% of the value shown in Figure 5. This demonstrates that this non-uniform plasma provides broadband operation for the beam-plasma interaction while still maintaining good gains per unit length which is in agreement with experiment $^4$.

Conclusions.

From this large signal analysis of beam-plasma interaction it is found that the fundamental and next two harmonics approach the theoretical maximum which can be achieved with perfect bunching under conditions where the velocity spread in the beam is still consistent with good efficiency operation. Also, an axially non-uniform plasma is seen to yield broad bandwidth. The results of this non-linear theory and calculations are useful in the evaluation of experimental tubes at saturated power levels, thus allowing a rapid exploration of the effect of various design parameters over a wider range than may be convenient experimentally and thereby assist in achieving optimum design.
Fig. 6. — Velocity of Discs at Some Values of $\beta_e z$.

References


