PULSE AMPLIFICATION IN NONLINEAR SYSTEMS

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It is shown that a system of weakly anharmonic oscillators can be used to amplify pulses. The conditions for amplification are given.

Recent measurements [1] have shown that it is possible to obtain strongly amplified echo pulses from a ferrimagnetic material. The purpose of this letter is to point out that pulse amplification is a general feature of a certain class of nonlinear systems, to outline the theory of amplification, and to show how parameters should be chosen to optimize the process.

Our treatment uses the same general ideas used in discussing echo phenomena [2]. We assume that the system consists of an ensemble of weakly nonlinear oscillators and we characterize each oscillator by its frequency ω and a generalized displacement $q[=q(\omega)]$. Let $G(\omega)\,\mathrm{d}\omega$ be the number of oscillators in the system with frequency between ω and $\omega + \mathrm{d}\omega$. We shall characterize the "width" of the oscillator distribution function by $\Delta\omega$ and take it to be much less than ω (frequency of the *system* is well defined). As in the case of echoes, two short pulses separated by the interval $\tau (\gg 1/\Delta\omega)$ are applied to the system. The first pulse represents the pulse to be amplified and the second pulse represents the "pump" or source of energy for the amplification process. For simplicity we shall assume that they are ideal impulses, but in reality they need only be wave trains with duration $\Delta t \ll 1/\Delta\omega$ and frequency approximately equal to that of the oscillators. Consequently, each oscillator of the system receives the same excitation by a pulse. We let the displacements resulting from the first and second pulses be q_1 and q_2, respectively. Normaly $q_1 \ll q_2$.

As to the *nonlinear* behavior of the oscillators, we suppose that their frequency is weakly dependent upon the excitation level, or more precisely, their energy, so that $\omega + \mathrm{d}\omega \approx \omega + (\partial\omega/\partial q^2)|q|^2$. Although we shall assume that frequency shift given by the second term is very small, it is important to note that the accumulated change in oscillator *phase* over the times of interest ($\sim \tau\mathrm{d}\omega$) is not, in general, small. In fact, precisely this change in phase is responsible for the amplified echo pulse. After the application of the first pulse at $t = 0$ the response of a single oscillator is* $q = q_1 \exp(\mathrm{i}\omega' t)$, with $\omega' = \omega + (\partial\omega/\partial q^2)q_1^2$. We calculate the total system response by summing over all oscillators,

*We use complex quantities; hereafter the real physical quantities are obtained by taking only the real part.

i.e., $Q = \int G(\omega) q_1 \exp(i\omega' t)\,d\omega = q_1 \exp(i\theta_1)\,g(t)$ where $g(t) = \int G(\omega)\exp(i\omega t)\,d\omega$ and $\theta_1 = (\partial\omega/\partial q^2)\,q_1^2 t$. Thus $g(t)$ is the small signal *impulse response* of the system which is just the Fourier tranform of the oscillator frequency distribution function. θ_1 is a (small) amplitude dependent phase change.

In a similar fashion, after the application of the second pulse at $t = \tau$, the response of an oscillator is
$q(t) = [q_1 \exp(i\omega'\tau) + q_2]\exp[i\omega''(t-\tau)]$ with

$$\omega'' = \omega + (\partial\omega/\partial q^2)[q_1^2 + q_2^2 + 2q_1q_2 \cos\omega'\tau] \qquad (1)$$

and the total system response is

$$Q = \int G(\omega)\,d\omega\,[q_1 \exp(i\omega'\tau) + q_2]\exp[i\omega''(t-\tau)] \quad (2)$$

Substituting eq. (1) into eq. (2) and using the identity

$$\exp(i\alpha\cos\phi) = \sum_{-\infty}^{\infty} (-i)^n J_n(\alpha)\exp(-in\phi),$$

we obtain

$$Q = \sum_{n=-\infty}^{\infty} A_{n-1}(t)\,g(t-n\tau) \qquad t > \tau \qquad (3)$$

with

$$A_{n-1}(t) = (-i)^n \exp(i\theta_2)[q_1 J_n(\alpha) + iq_2 J_{n-1}(\alpha)] \quad (4)$$

and $\alpha = 2(\partial\omega/\partial q^2)\,q_1 q_2(t-\tau)$. We have assumed $\theta \ll 1$ (weak first pulse).

Eq. (3) indicates that the system response consists of a series of pulses occuring at $t = \tau$, 2τ, 3τ, ..., each having a shape given by the system impulse response. The pulse at $t = \tau$ may be regarded as the response to the second pulse, whereas subsequent pulses are echo pulses. Provided $\Delta\omega \gg 1/\tau$ the various pulses do not overlap significantly in time. We are concerned primarily with the first echo pulse, i.e., amplification. The amplitude of the first echo pulse is given by $A_1(2\tau)$, since near $t = 2\tau$ only the second term in eq. (3) contributes significantly. Furthermore in eq. (4) $q_1 J_2(\alpha) \ll q_2 J_1(\alpha)$ so that

$$|A_1| \approx q_2 J_1(\alpha) = 2q_1 q_2^2\,(\partial\omega/\partial q^2)\,\tau, \qquad \alpha \ll 1 \quad (5)$$

Several important conclusions can be drawn from this result. First, for sufficiently small α (sufficiently small first pulse, or second pulse) the echo amplitude is *linear* in the first pulse amplitude. Hence superposition applies with respect to the first pulse ‡. Defining the gain κ of the system to be the ratio of the first echo response to the response of the system to the initial pulse, we find from eq. (5) that the small signal gain is $\kappa = 2(\partial\omega/\partial q^2)\,q_2^2$, which is just twice the amplitude-dependent phase change in the interval τ which is produced by the second pulse. It can be made much larger than unity by the appropriate choice of q_2 or τ ##. Note also that in this linear regime the second echo pulse in *much smaller* than the first echo pulse and its amplitude is proportional to q_1^2. A second important conclusion is that in no event can the amplified pulse be greater than 0.58 times the response to the second (pump) pulse, since the Bessel function $J_1(\alpha)$ takes on a maximum value of 0.58 for $\alpha = 1.84$. This corresponds to the saturation output of the amplifier and only for input pulses well below the value corresponding to $\alpha = 1.84$ is it linear.

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‡ However, the echo pulse is inverted in time, i.e., the leading edge of the first echo corresponds to the trailing edge of the original pulse. Application of a third (pump) pulse T seconds after the second pulse will cause an "echo of the echo" which is not inverted and is delayed by $2T$ seconds from the original pulse.

However, τ must be smaller than the collisional relaxation time.

References

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