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PHASE AND GROUP VELOCITY IN AN ANISOTROPIC PLASMA

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I. Introduction

We discuss the relationship between phase and group velocities of electromagnetic waves in a magnetoplasma. In general, they are not parallel and when the index of refraction exhibits a resonance ( $n = c/v_p \rightarrow \infty$ ) they are nearly perpendicular. When determining whether propagation between two remote points is possible, the allowed directions of the group velocity vector are important and the usual phase velocity plots are misleading.

II. Phase Velocity

We first outline the dispersion of plane harmonic waves (wave fields  $\exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$ ) in a plasma in a magnetic field  $\mathbf{E}_0 = B_0 \hat{z}$  in the  $z$  direction. Maxwell's curl equations are  $\mathbf{k} \times \mathbf{E} = \omega \mathbf{E}$  and  $\mathbf{k} \times \mathbf{H} = -(\omega/c^2) \mathbf{K} \cdot \mathbf{E}$  where  $\mathbf{K} = K(\omega, \mathbf{k})$  is the tensor dielectric function of the plasma. For simplicity we limit our discussion to sufficiently large static magnetic fields so that, effectively, electrons move along field lines, i.e.,

$$\mathbf{K} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K \end{pmatrix} \quad (1)$$

where  $K = K(\omega, k_z) = 1 + \chi(\omega, k_z)$  and  $\chi$  is the susceptibility. For a cold collisionless plasma  $\chi = -\omega_p^2/\omega^2$  where  $\omega_p = (ne^2/\epsilon_0 m)^{1/2}$  is the plasma frequency.

For plane waves whose wave vector (and hence phase velocity) makes an angle  $\psi$  with respect to the magnetic field,  $\mathbf{n} = \mathbf{k}c/\omega = (n \sin \psi, 0, n \cos \psi)$  and the (dispersion) equation for the index of refraction  $n$  is readily found to be (with  $\psi = \cos \psi$ ).

$$1 - n^2 + \chi - \chi n^2 \mu^2 = 0 \quad (2)$$

The solution of this equation is

$$n^2 = (\epsilon/v_p)^2 = (1 + \chi)/(1 + \chi \mu^2) \quad (3)$$

When  $\omega_p^2/\omega^2 = -\chi < 1$ ,  $n$  is real for all angles  $\psi$ , but when  $\omega_p^2/\omega^2 = -\chi > 1$ ,  $n^2 = \infty$  (resonance) for  $\mu^2 = \mu_c^2 = -1/\chi$  i.e., when  $\cos \psi = \cos \psi_c = \omega/\omega_p$ . Furthermore, when  $\omega > \omega_c$ , i.e.,  $\psi < \psi_c$ ,  $n$  is real (propagating waves) whereas when  $\omega < \omega_c$ , i.e.,  $\psi > \psi_c$ ,  $n$  is imaginary (evanescent waves). This behavior is illustrated in the right half of Fig. 1, which is a polar plot of  $v_p/c = 1/n$  for  $\omega/\omega_p = 2$  or  $\psi_c = 60^\circ$  (the dots correspond to  $5^\circ$  increments in  $\psi$ ). The angle  $\psi_c$  defines a cone, whose axis coincides with the magnetic field, within which the phase velocity vector must lie and on which it tends to zero. We refer to this as the phase velocity cone.

III. Group Velocity

For each  $(\omega, \mathbf{k})$  we can associate a group velocity given by  $\mathbf{v}_g = \nabla_{\mathbf{k}}[\omega(\mathbf{k})]$ . Using (2) we can readily obtain expressions for the magnitude of the group velocity and the angle  $\theta$  which it makes with the magnetic field

$$(v_g/c)^2 = \frac{(1 + \chi)(1 + \chi \mu^2)}{(1 + 2\chi \mu^2 + \chi^2 \mu^4)} \quad (4)$$

$$\tan \theta = -\tan \psi / (1 + \chi) \quad (5)$$

Thus the group velocity is not in the same direction as the phase velocity. In fact the component of  $\mathbf{v}_g$  perpendicular to  $\mathbf{B}_0$  is opposite to the perpendicular component of  $\mathbf{v}_p$ . Near resonance ( $n^2 \rightarrow \infty$ )  $\mathbf{v}_g$  and  $\mathbf{v}_p$  are perpendicular (and coplanar with  $\mathbf{B}_0$ ). This behavior is illustrated in the left half of Fig. 1, which is a polar plot of  $v_g/c$ . Each dot on the group velocity plot is to be associated with the corresponding dot on the phase velocity plot (e.g., A, B, C, ...). Equation (5) shows that  $\theta$  increases monotonically with  $\psi$  and reaches its maximum value  $\theta_c$  at resonance ( $\psi = \psi_c$ ). Furthermore  $\psi_c + \theta_c = 90^\circ$ . Thus  $\theta_c$  defines a second cone, within which the group velocity vector must lie and on which it tends to zero. We refer to this cone as the group velocity cone. Since propagation between two remote points is determined by the group velocity, we wish to emphasize the importance of  $\mathbf{v}_g$  and hence the group velocity cone rather than the customary phase velocity plots and cone.

For the case studied here ( $B_0 \neq 0$ )  $\psi_c$  decreases from  $90^\circ$  to  $0^\circ$ , whereas  $\theta_c$  increases from  $0^\circ$  to  $90^\circ$  when  $\omega/\omega_p$  increases from 0 to 1. When  $\omega/\omega_p > 1$  the cones do not exist.

IV. Effect of Electron Thermal Velocities

The behavior near resonance is modified substantially by the inclusion of electron thermal velocities, since  $\chi = -\omega_p^2/(\omega^2 - k_z^2 v_{th}^2)$  with  $v_{th}^2 = 3kT_e/m \equiv \beta^2 c^2$ . Equation (2) becomes

$$n^4 (\mu_c^2 \mu^2 \beta^2) + n^2 (\mu^2 - \mu_c^2 - \mu_c^2 \mu^2 \beta^2) - (1 - \mu_c^2) = 0 \quad (6)$$

When  $n^2 \ll 1/\beta^2$ , Eq. (3) is recovered, and when  $n^2 \gg 1/\beta^2$

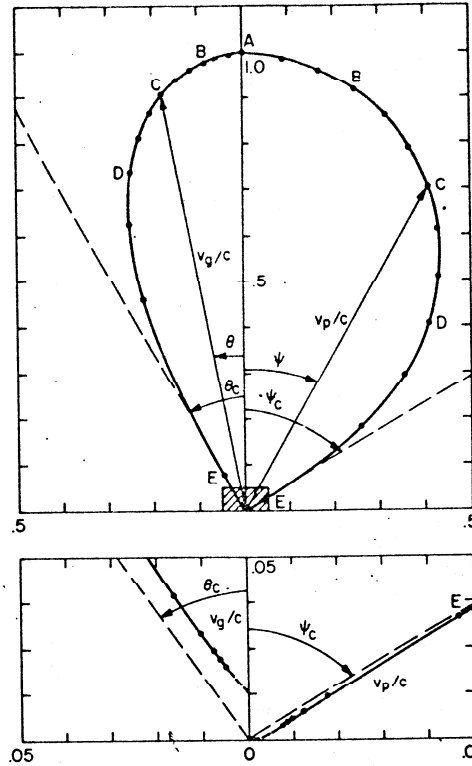


Figure 1. Polar plot of phase velocity (right side) and group velocity (left side) for  $B_0 \neq 0$ ,  $\omega_p/\omega = 2$ . Lower plot shows shaded region, where electron thermal velocities are important, on an expanded scale.

$$n^2 = \frac{\mu^2 - \mu_c^2}{\mu_c^2 \mu^2 \beta^2} \quad (7)$$

The modification in the polar phase and group velocity plots are shown in the lower part of Fig. 1 for  $\beta = .01$ . (Dots correspond to  $1^\circ$  increments). We see that inclusion of electron thermal velocity effects now permits all values of  $\psi$  but  $\theta$  is nevertheless restricted to  $\theta < \theta_c$  as before. We note also that the  $v_p$  is a single valued function of  $\psi$  although  $v_g$  is a double-valued function of  $\theta$  (one fast wave  $v_g \sim v_c$  and one slow wave  $v_g \sim \beta c$ ). It should also be noted that when  $\chi$  is determined from the Vlasov equation, the slow waves exhibit strong Landau damping in the dashed part of Fig. 1.

V. Conclusion

By considering a special case ( $B_0 \neq 0$ ) we conclude that in an anisotropic magnetoplasma the phase and group velocities are, in general, in different directions and that the allowed directions of these two vectors are contained within two different cones, the phase velocity cone ( $\psi < \psi_c$ ) and the group velocity cone ( $\theta < \theta_c$ ), on which  $v_p \neq 0$  and  $v_g \neq 0$ . The cone angles are complementary, i.e.,  $\psi_c + \theta_c = 90^\circ$ . While the phase velocity diagrams are most frequently discussed, the group velocity diagrams are relevant when determining whether transmission between two spatially separated points is possible. This is further confirmed by the fact that the time average Poynting vector of a dipole antenna vanishes outside of the group velocity cone.

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References

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