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ON THE CONNECTION BETWEEN RESONANCE CONES AND GUIDED MODES IN AN ANISOTROPIC PLASMA*

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Abstract

In an anisotropic plasma, where the perpendicular and parallel elements of the cold plasma dielectric tensor ($K_{\perp}$ and $K_{\parallel}$) have opposite signs, the equation describing the propagation of electrostatic disturbances is hyperbolic and disturbances propagate along characteristics. This gives rise to the "resonance cone" phenomenon when the source is spatially localized. On the other hand, plasma columns and slabs are known to support guided-wave modes under the same conditions. In this paper we explore the relationship between these two rather differently appearing phenomena, show that electrostatic disturbances from a localized source in a plasma slab can be thought of <u>either</u> as a series of multiply reflected (from the slab boundaries) resonance cones, <u>or</u> as a superposition of guided-wave modes. Depending on the source size and importance of dissipation, one of the other points of view may be more useful.

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I. Introduction

The resonance cone phenomenon[1] first arose in connection with a study of the fields of a localized source (small antenna) in a highly anisotropic plasma. Because disturbance can propagate in certain preferred directions with respect to the magnetic field without appreciable spreading, the resonance cone phenomenon can be of great importance in plasma heating[2] for thermonuclear reactors and in the propagation of very low frequencies in the ionosphere and magnetosphere. Resonance cones occur when $K_{\parallel}K_{\perp} < 0$ ($K_{\parallel}$ and $K_{\perp}$ are the elements of the cold plasma dielectric tensor parallel and perpendicular to the static magnetic field. In this regime the equation[3] governing the propagation of two-dimensional electrostatic disturbances,

$$\nabla \cdot (\underline{\underline{K}} \cdot \nabla\phi) = \frac{\partial}{\partial x} \left(K \frac{\partial\phi}{\partial x}\right) + K \frac{\partial^2\phi}{\partial z^2} = 0 \tag{1}$$

is hyperbolic rather than elliptic. Disturbances propagate along characteristics which make an angle

$$\theta_c = \tan^{-1}(-K_{\perp}/K_{\parallel})^{1/2}$$

with respect to the static magnetic field. Disturbances radiating from spatially localized sources propagate in these preferred directions.

Bounded plasmas or, more specifically, plasma columns or slabs, are known to support guided-wave modes[4]. These Trivelpiece-Gould modes are important in understanding a variety of phenomena[5] in plasma columns (Q-machines, etc.). A plasma column can support a spectrum of such modes, with spatial variations perpendicular to the cylindrical axis (and magnetic field) of each characterized by a different transverse eigenfunction and by a different parallel wave number, k_n . These modes exist in the same parameter regime as for existence of the resonance cones, namely $K_\parallel K_\perp < 0$.

The purpose of this paper is to explore these two apparently unrelated phenomena which occur for the same parameter regime and show (a) that they are, in fact, different manifestations of the same equations, (b) how both points of view can be used to describe the same phenomena, but that (c) in most circumstances one point of view may be preferable (more useful) than the other.

II. Guided Modes of a Plasma Slab

For purposes of exposition we consider the one-dimensional slab geometry of Figure 1 and make some important simplifying assumptions:

a) plasma spatially uniform with density falling abruptly to zero at the perfectly conducting metal wall;

b) cold plasma model, neglecting ion motions.

From the second it follows that the plasma is completely characterized by its dielectric tensor, of which only

$$K_\perp = 1 - \omega_{pe}^2/(\omega^2 - \omega_{ce}^2)$$

$$K_\parallel = 1 - \omega_{pe}^2/\omega^2$$

are important. From (1) it readily follows that guided-wave solutions satisfying the conducting wall boundary conditions at $x = 0,a$ are

$$\phi_n = \sin \frac{n\pi x}{a} e^{ik_n z}$$

where

$$k_n = (\frac{n\pi}{a})(-K_\perp/K_{\parallel})^{1/2} \qquad n = \pm 1, \pm 2, \cdots \qquad (2)$$

The quantity $(-K_\perp/K_{\parallel})^{1/2}$ occurs repeatedly, so we denote it by α . Furthermore, neglecting ion effects and collisions

$$\alpha^2 = -\frac{K_\perp}{K_{\parallel}} = \frac{\omega^2(\omega_{UH}^2 - \omega^2)}{(\omega_{ce}^2 - \omega^2)(\omega_{pe}^2 - \omega^2)}$$

Figure 2 summarizes the characteristics of the lowest few modes for the case $\omega_{ce} > \omega_{pe}$. When $\omega_{ce} < \omega_{pe}$ the roles of ω_{pe} and ω_{ce} in this diagram are simply interchanged.

III. Excitation of Guided Modes

We now show that the potential produced by the source in Figure 1 can be represented as a superposition of the Trivelpiece-Gould guided modes. The Fourier transform of (1) with respect to z is

$$\frac{d^2\Phi(x,k)}{dx^2} + \frac{k^2}{\alpha^2}\Phi_k(x,k) = 0$$

A proper solution of (8) which satisfies the boundary conditions at $x = 0$ and a is

$$\Phi(x,k) = \Phi(0,k)\,\frac{e^{ikx/\alpha} - e^{-ik(x-2a)/\alpha}}{1 - e^{2ika/\alpha}} \qquad (3)$$

where

$$\Phi(0,k) = \int_{-\infty}^{\infty} \phi(0,z)\, e^{-ikz}\, dz$$

is the Fourier transform of the potential along the lower plate. The latter is presumed to be known.

Rewriting the inverse Fourier transform of (9) slightly, we have

$$\phi(x,z) = \int_{-\infty}^{\infty} \Phi(0,k) \frac{\sin k(a-x)/\alpha}{\sin ka/\alpha} e^{ikz} \frac{dk}{2\pi}$$

which has simple poles along the real k axis[6] at

$$k = k_n = n\pi\alpha/a \qquad n = \pm1, \pm2, \cdots$$

To determine how to deform the contour around the poles we note that any small dissipation (collisions) will give both $K_\perp$ and $K_{\parallel}$ a small positive imaginary part. This, in turn, causes α to have a small imaginary part ($\alpha = \alpha' + i\alpha''$) which is positive on the lower branch of Figure 2 and negative on the upper branch. We consider the former case in detail. The simple poles are displaced from the real axis as illustrated in Figure 3. Furthermore, for positive z we can close the contour above, since this makes no contribution[7]. Using the residue theorem we obtain

$$\phi(x,z) = i\alpha a \sum_{n=1}^{\infty} \Phi(0,k_n) \sin n\pi x/a \, e^{ik_n|z|}$$

$$+ \text{ contribution from singularities of } \Phi(0,k) \qquad (4)$$

where k_n is given by Eq. (6). We expect the contributions from the singularities in $\Phi(0,k)$ to decay rapidly in z and thus become important. The remaining series expresses the potential as a superposition of undamped (when $\alpha'' = 0$) Trivelpiece-Gould modes. To be specific, if we assume that $\phi(0,z) = \frac{w/\pi}{z^2 + w^2}$, i.e., that the source potential has a half width 2w, it is readily shown that the Fourier transform is

$\Phi(0,k) = e^{-|k|w}$ and that equation (12) becomes

$$\phi(x,z) = i\alpha a \sum_{n=1}^{\infty} e^{-k_n w} \sin \frac{n\pi x}{a} \; e^{-ik_n z} \tag{5}$$

with k_n given by Eq. (2). The first factor, $e^{-k_n w}$, shows that the amplitudes of the modes decrease with increasing mode number and that the decrease is more rapid for larger w . In the limit $w \to 0$, the source becomes $\delta(z)$ and the mode amplitudes do not diminish. In this limit we obtain an expression for the Green's function for an arbitrary potential distribution in the plane $x = 0$,

$$G(z,z') = -i\alpha a \sum_{n=1}^{\infty} \sin n\pi x/a \;\; e^{ik_n|z-z'|}$$

IV. Resonance Cones

If the source in Figure 1 is localized very near to the region $z = 0$ then we expect, because of the hyperbolic character of Eq. (1) when $K_{\parallel} K_{\perp} < 0$, the disturbance to be concentrated along certain directions given by Eq. (2). We also expect, possibly, multiple reflections of the resonance cones from the metallic conductors, as shown in Figure 4. To show that this is indeed the case, we evaluate the inverse Fourier transform of (1) using a different method. For $k < 0$, $|e^{2ika/\alpha}| < 1$ when $\text{Im}\,\alpha < 0$ and we can write for the denominator of (3) $[1 - \exp(2ika/\alpha)]^{-1} = \sum_{n=0}^{\infty} \exp(2ikan/\alpha)$. For $k < 0$ we multiply the numerator and denominator of Eq. (3) by $\exp(-2ika/\alpha)$ and employ a similar expansion. The potential can therefore be expressed

$$\phi = - \sum_{n=0}^{\infty} \int_{-\infty}^{0} \Phi(0,k)\, e^{ikz} [e^{ik(x-2a)/\alpha} - e^{-ikx/\alpha}]\, e^{-2nika/\alpha}$$

$$\sum_{n=0}^{\infty} \int_{0}^{\infty} \Phi(0,k)\, e^{ikz}\, [e^{ikx/\alpha} - e^{-ik(x-2a)/\alpha}]\, e^{2nika/\alpha}$$

which can be written

$$\begin{aligned} \phi = & \sum_{n=0}^{\infty} f(z + x/\alpha + 2na/\alpha) + \sum_{n=0} f(-z + x/\alpha + 2na/\alpha) \\ & - \sum_{n=0}^{\infty} f(z - x/\alpha + 2a/\alpha + 2na/\alpha) - \sum_{n=0} f(-z - x/\alpha + 2a/\alpha + 2na/\alpha) \end{aligned} \tag{6}$$

where

$$f(\zeta) = \int_{0}^{\infty} \phi(0,k)\, e^{ik\zeta} \frac{dk}{2\pi} \tag{7}$$

Making the earlier assumption about the source, $\phi(0,z) = w/\pi(z^2+w^2)$ we obtain

$$f(\zeta) = \frac{1}{2\pi(w-i\zeta)}$$

We note that as the source becomes more localized ($w \to 0$), $f(\zeta)$ becomes more sharply peaked and, in the limit $w \to 0$,

$$f(\zeta) = \frac{1}{2}\,\delta(\zeta) + \frac{i}{2\pi\zeta}$$

Equation (6) can readily be seen to represent multiply reflected resonance cones as illustrated in Figure 4. The first terms in the first and second series describing the disturbances emanating directly from the source ($z = 0$), the first terms of the third and fourth series representing the first reflection of the disturbance from the upper conductor, the second terms in the first and second series describing the second reflection of the disturbance from the lower conductor, etc. In the absence of dissipation and boundary effects not described by our model, the "shape" of the disturbance is preserved upon reflection, although there is a change in the sign of the potential.

V. Transition from Resonance Cones to Guided Modes when Dissipation is Present

We have seen that the potential, and consequently the fields, can be expressed either as a superposition of multiply reflected resonance cones or as a superposition of the Trivelpiece-Gould guided modes. In the latter case the number of guided modes required to properly represent the source becomes very large when the source is highly localized (w small) and the former representation may be more convenient. If we further include collisional dissipation the solution can be shown to have a resonance-cone-like behavior near the source, with the higher order guided modes decaying most rapidly, leaving only the lowest (n=0) mode far from the source. To see this we recall that the inclusion of collisions causes both $K_\perp$ and $K_{\parallel}$, as well as α to acquire imaginary parts. The expression for k_n in Eqs. (2) becomes

$$k_n = \frac{n\pi}{a}(\alpha' + i\alpha'')$$

where α' and α'' denote the real and imaginary parts of α. Thus the damping per unit length ($\sim \text{Im } k_n$) increases with mode number and the higher modes decay most rapidly leaving, far from the source, only the lowest mode. This is shown graphically in Figure 5 where $|\phi(x,z)|$ is plotted for a narrow (w = 0) gap source excitation. Near the source one sees a sharply peaked disturbance emanating from the gap, reflected alternately from the two conductors with one peak slowly eroding so that after about four reflections it is mainly the lowest (n=1) guided mode which remains.

Footnotes and References

1. H. Kuehl, Phys. Fluids 5, 1095 (1962);
   R. K. Fisher and R. W. Gould, Phys. Rev. Lett. 22, 1093 (1969) and Phys. Fluids 14, 857 (1971);
   N. Singh and R. W. Gould, Phys. Fluids 16, 75 (1973);
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2. R. J. Briggs and R. R. Parker, Phys. Rev. Lett. 29, 852 (1972).

3. For simplicity we continue this discussion in slab geometry. The generalization to other geometries is straightforward. We also use a cold plasma model with sinusoidal ($e^{-i\omega t}$) time dependence.

4. A. W. Trivelpiece and R. W. Gould, J. Appl. Phys. 30, 1784 (1959).

5. T. K. Chu et al, Phys. Rev. Lett. 31, 211 (1973);
   R.P.H. Chang and M. Porkolab, Phys. Rev. Lett. 32, 1227 (1974);
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6. $\Phi(0,k)$ may also have singularities in the complex k-plane, but normally not on the real axis.

7. Provided $\Phi(0,k)$ tends to zero sufficiently rapidly at infinity.

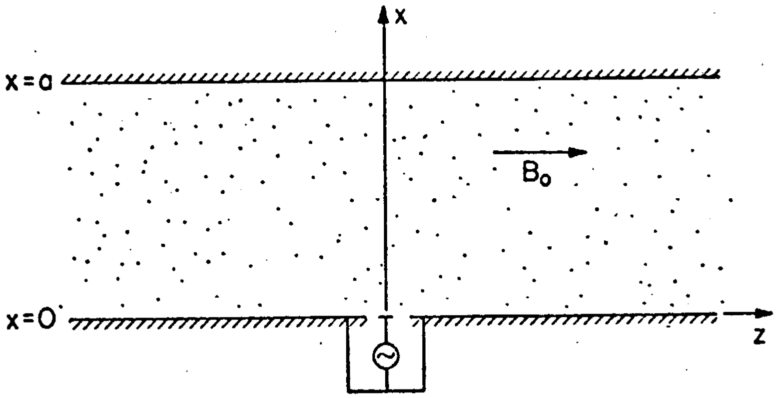

Figure 1.

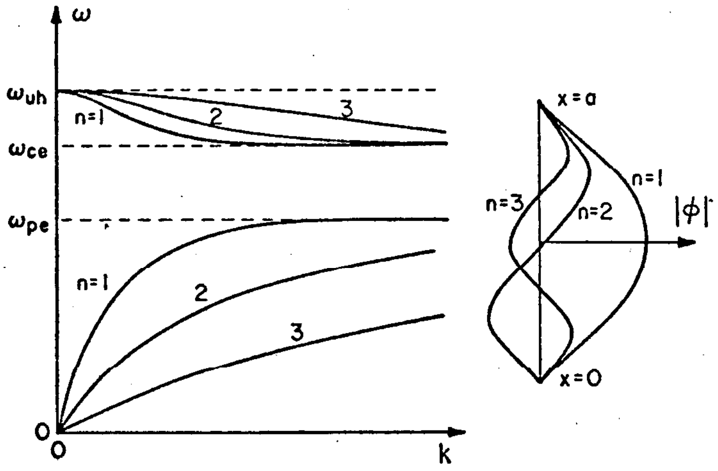

Figure 2.

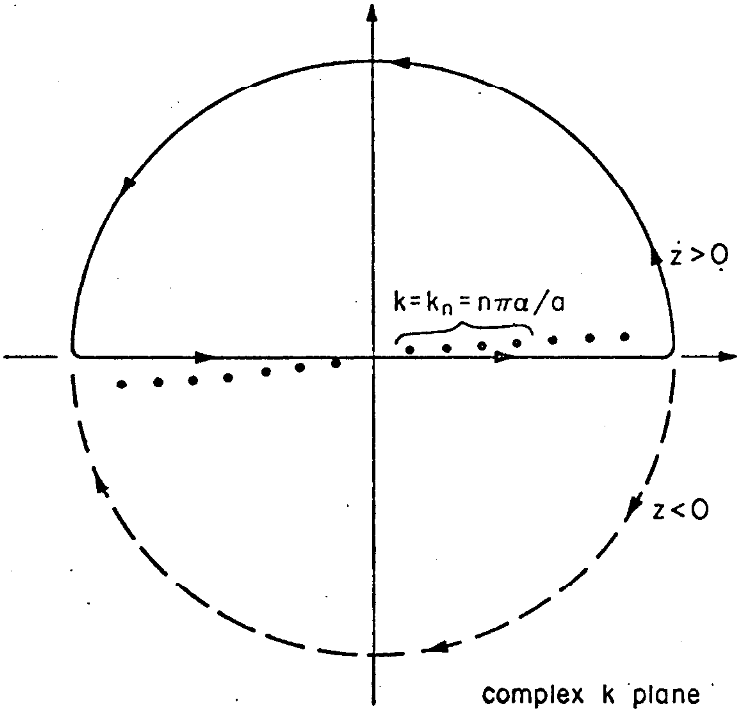

Figure 3.

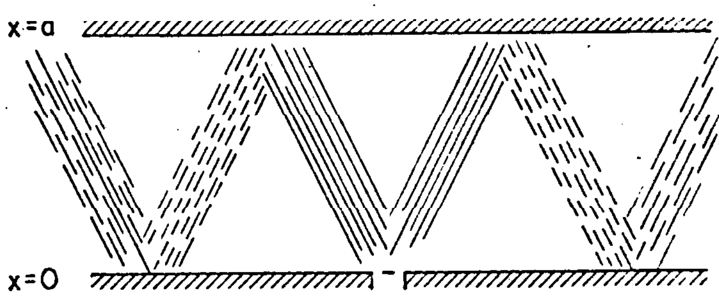

Figure 4.

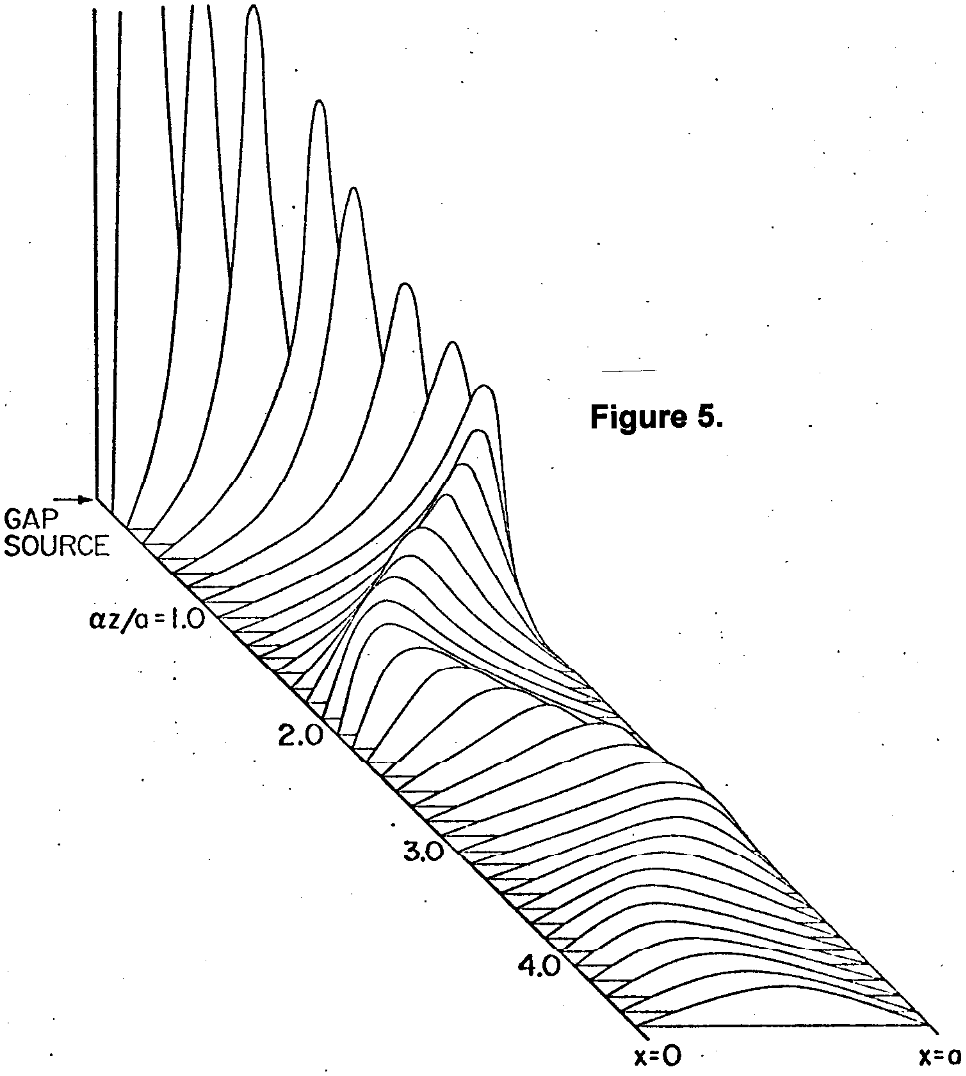

Figure 5.