Wave Angular Momentum in Nonneutral Plasmas*

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Abstract. Angular momentum and energy are added (or removed) when exciting a mode, such as a diocotron, Trivelpiece-Gould, or Dubin mode, and we discuss rates at which mode angular momentum and energy are added by applied fields. Excitation of a plasma mode is an effective way to transfer angular momentum and energy to the plasma because it is a resonant process. We relate this to recent experiments on compression and expansion of plasmas using a "rotating wall" field. We also calculate the torque on a Coulomb *crystal* which is phase-locked to a "rotating wall" field and describe phase oscillations and the maximum rate of acceleration which can be achieved.

Early experiments(1) showed that asymmetric applied potentials can cause particle transport in nonneutral plasmas, and static field errors are thought to be responsible for the anomalous loss of particles from traps. Collective modes, either time dependent or static, can enhance the asymmetric fields responsible for this transport, and either inward or outward transport can occur. Recent "rotating-wall" field experiments(2) have brought some of these ideas into sharper focus by showing that nonneutral plasmas can be contained indefinitely with such fields and that the angular momentum transfer rates are much larger when the excitation frequency corresponds to one of the Trivelpiece-Gould(TG) mode frequencies.

Theoretical attempts to understand transport have generally focussed on the details of *particle* transport near resonant surfaces in the plasma(3). In this paper, we focus instead on the angular momentum and energy added when a *mode* is excited. The added angular momentum and energy is associated with a coherent wave perturbation in the plasma. If the mode damps, the added wave angular momentum and energy then become part of the equilibrium plasma. Dissipative processes are required for damping and the details of these processes are important in determining exactly where the momentum and energy is deposited within the plasma. However, it is useful to separate the process of transferring angular momentum to the wave from its redistribution within the plasma and to obtain transfer rates from properties of the

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modes. It is not actually necessary for the applied field to be a "rotating-wall" field, so long as the mode excited has a rotating field. We obtain the transfer rates from a susceptiblity, χ, which is the ratio of oscillating charge induced on the wall to the oscillating applied potential which excites the mode.

A spatially uniform cold electron plasma has a canonical angular momentum $P_\theta = Nm(\omega_r - \omega_c/2) < r^2 >$, where N is the total number of electrons of mass m, ω_r is the rotation frequency, independent of r for a spatially uniform plasma, and $\omega_c = eB_o/m$ is the electron cyclotron frequency and $< r^2 >$ is the mean square radius of the plasma fluid. The first term represents the mechanical part of the canonical angular momentum and the second term represents the magnetic part. For plasmas obeying the drift approximation ($\omega_r \ll \omega_c/2$), the latter term is larger than the former, so that the canonical angular momentum is actually negative. Thus a positive torque, which increases the canonical angular momentum, decreases the *magnitude* of the angular momentum and therefore $< r^2 >$, thereby compressing the plasma. However, when $\omega_r > \omega_c/2$, $P_\theta > 0$ and a positive torque expands the plasma. While we discuss an electron plasma here, these ideas and results are applicable to ion plasmas with appropriate changes in sign of various quantities.

We can calculate the torque T on the plasma either by integrating the moment of the electric force over the volume V of the plasma, or by integrating the stress tensor over a surface S (the negative of the torque on the wall charges) outside the plasma at the wall. We can also calculate P, the power input to the plasma, from the wall potential times the inward displacement current at the wall:

$$T = -\int_V rneE_\theta dV = \epsilon_o \int_S E_r E_\theta \, b \, dS = \frac{1}{2} Re[imE_{rmk}\phi^*_{mk}]S$$

$$P = -\epsilon_o \int_S \frac{\partial E_r}{\partial t} \phi \, dS = \frac{1}{2} Re[i\omega E_{rmk}\phi^*_{mk}]S.$$

where $S = 2\pi bL$ is cylindrical surface at $r = b$, the wall and length L, so that $k = k_n = n\pi/L$ (periodic boundary conditions). The wave torque involves quadratic wave quantities, such as the field E_θ times the perturbed E_r of the wave, $\sim exp[i(m\theta + kz - \omega t)$. ϕ_{mk} and E_{rmk} are the complex amplitudes of the oscillating wave potential and radial electric fields at the wall, respectively. Such fields travel with angular velocity ω/m. We regard ϕ_{mk} as applied and E_{rmk} as the response and define the response function, or susceptiblity, for a single wave with wave number pair (m, k) as

$$\chi_{mk}(\omega) = -\left. \frac{rE_{rmk}}{\phi_{mk}} \right|_{r=b}$$

This allows us to write the torque and power input in terms of the applied potential as

$$T = -\frac{1}{2} Re[im\chi_{mk}(\omega)\phi_{mk}\phi^*_{mk}]S$$

$$P = -\frac{1}{2} Re[i\omega\chi_{mk}(\omega)\phi_{mk}\phi^*_{mk}]S.$$

It follows from these relations that $P/T = \omega/m$, the angular velocity of the rotating field. Both P and T are quadratic in the applied potential.

We anticipate that $\chi_{mk}(\omega)$ will have poles at the mode frequencies, with one pole for each *radial* eigenmode (whose number is denoted by l), and to be of the form,

$$\chi_{mk}(\omega) = \sum_l \frac{R_{lmk}}{(\omega - \omega_{lmk} + i\gamma_{lmk})} + \text{other terms}$$

ω_{lmk}, R_{lmk}, and γ_{lmk} are the frequency, residue or coupling strength, and damping rate of the mode with radial, azimuthal, and axial mode numbers (l, m, k). These are measurable quantities. If we focus on just one mode, hence one term in this series, this is the classic case of an oscillator driven by a sine wave. For a dissipationless plasma, $\gamma_{lmk} = 0$, and the susceptibility function will be purely real, with poles at the mode frequencies. Then there is continuous increase, or decrease, of angular momentum and energy of the mode only if the applied frequency is *exactly* equal to a mode frequency. With damping, the *steady* input rate at resonance is proportional to R_{lmk}/γ_{lmk}.

The *sign* of R_{lmk} determines the direction of the transfer. When R_{lmk} is negative, transfer is to the plasma, and when R_{lmk} is positive, transfer is from the plasma. When $\gamma_{lmk} = 0$ and at exact resonance, the oscillator is continuously excited until the pulse ends. Off resonance, there is transfer to and from the mode at the *difference* frequency with a net transfer to the plasma if $R_{lmk} < 0$ and from the plasma if $R_{lmk} > 0$. For an applied *pulse* whose length is short compared to the beat period there will be a momentum transfer to or from the plasma, depending on the sign of R_{lmk}, so long as the pulse has frequency components at one of the mode frequencies.

We note that the input admittance of a patch electrode can be written in the form

$$Y(\omega) = -i\omega\epsilon_o \frac{A}{b} \sum_{mn} \left|S_{mn}\right|^2 \chi_{mn}(\omega),$$

where S_{mn} is a "structure" factor for the patch with $k = n\pi/L$, and A is its area. This includes all of the modes and is a measureable quantity. In this connection we again note it is not necessary for the applied field to be rotating in order to excite a mode which rotates.

To obtain $\chi_{mk}(\omega)$ one must solve the potential equation within the plasma and surrounding vacuum regions for a sinusoidally varying potential applied to the wall electrode. $\chi_{mk}(\omega)$ is simply the negative of the logarthmic derivative of Φ at the wall. For this discussion, we consider a cold uniform plasma cylinder of radius a. The potential equation is $\nabla \cdot \boldsymbol{\epsilon} \cdot \nabla\Phi = 0$, where $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(\omega)$ is the linear dielectric tensor for sinusoidally varying fields. The methods for solving this equation when $\Phi = 0$ at the wall $(r = b)$, and the for modes which result, have been discussed extensively in the literature(4,5). This model has $\gamma_{lmk} = 0$. It is straightforward to obtain the solution when there is an applied time-varying potential applied on the wall, and to obtain

$$\chi_{mk}(\omega) = G_3 \frac{F(\omega)+G_1}{F(\omega)+G_2}$$

$$F(\omega) = \epsilon_1(\omega) Ta [\frac{J_m'(Ta)}{J_m(Ta)}] + \epsilon_2(\omega), \quad G_1 = ka \frac{I_m'(kb)K_m'(ka) - K_m'(kb)I_m'(ka)}{K_m'(kb)I_m(ka) - I_m'(kb)K_m(ka)},$$
$$G_2 = ka \frac{I_m(kb)K_m'(ka) - K_m(kb)I_m'(ka)}{K_m(kc)I_m(ka) - I_m(kc)K_m(ka)}, \quad G_3 = kb \frac{K_m'(kb)I_m(ka) - I_m'(kb)K_m(ka)}{K_m(kb)I_m(ka) - I_m(kb)K_m(ka)},$$

with $T^2 = -k^2\epsilon_3(\omega)/\epsilon_1(\omega)$. Primes denote derivatives with respect to the argument. $F(\omega)$ contains all of the frequency dependent terms and properties of the plasma and G_1, G_2 and G_3 depend only on the wave numbers m and k and the dimensions a and b. The components of the dielectric tensor are given in Ref. 5. The poles of $\chi_{km}(\omega)$ give the mode frequencies and are obtained from the solution of $F(\omega) + G_2$=0. $F(\omega)$, and thus $\chi_{km}(\omega)$, has alternating poles and zeros for real frequencies because of the Bessel functions. They are clustered about about $\omega = m\omega_r$ for small ka, becoming very dense at this frequency. Since m and $k = n\pi a/L$ are fixed, these various poles correspond to higher order *radial* modes, with increasing number of radial nodes, characterized by radial mode number l. There is also one isolated mode, the *diocotron* mode with $\omega = \omega_r[1-(a/b)^{2m}]$ for $ka = 0$. Coupling to the latter is quite large.

It is straightforward to obtain the frequencies and residues for each of these poles numerically. Illustrative results are shown in Figures 1a and 1b, respectively. Both are in units of ω_r, the rotation frequency. Only the lowest few T-G modes and the one

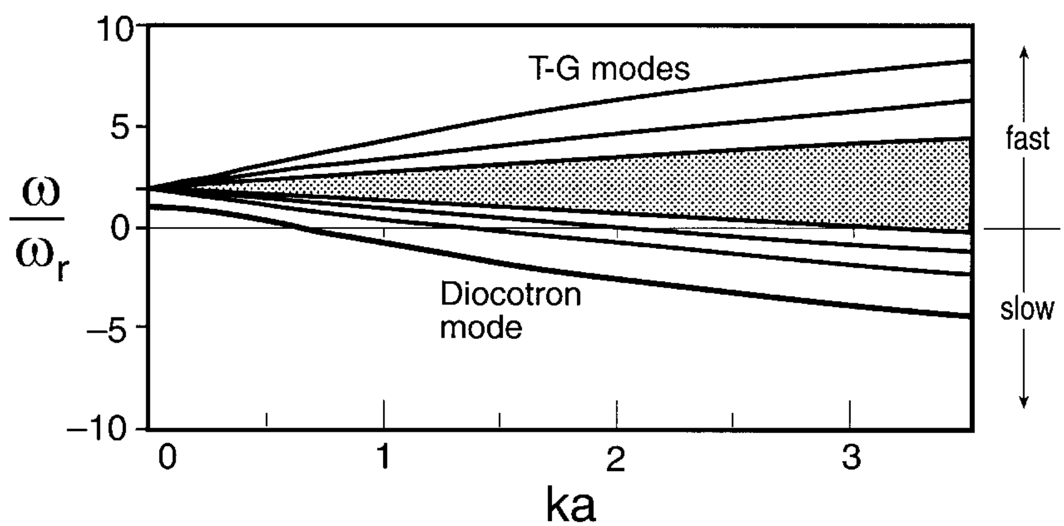

FIGURE 1a. Mode Frequencies in units of ω_r versus ka for $m = 2,\ b/a = 2,\ \omega_p/\omega_r = 10$.

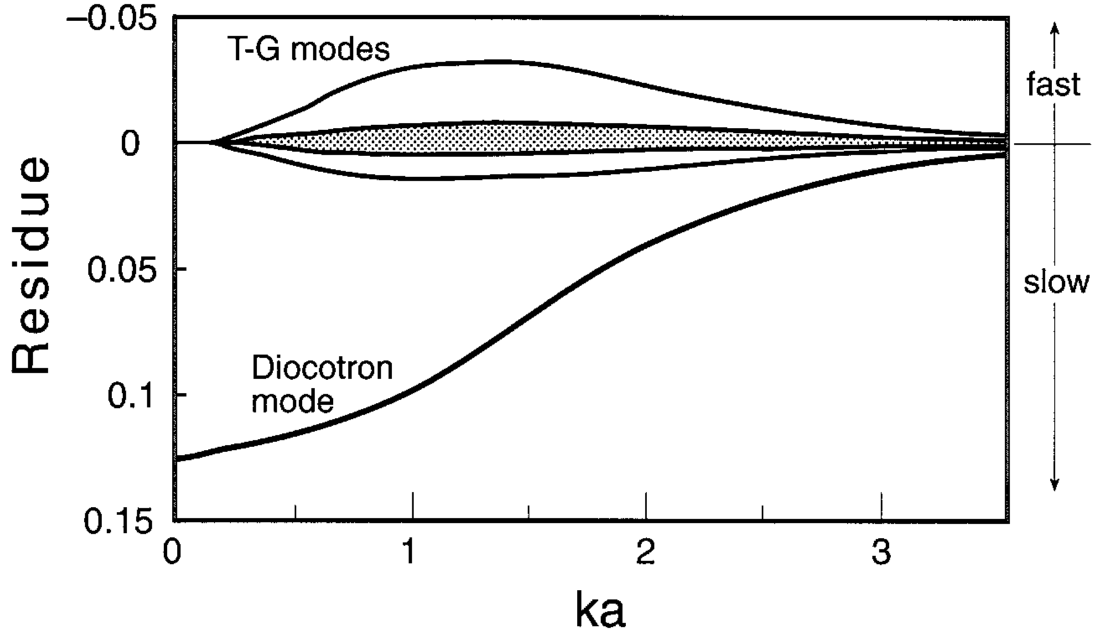

FIGURE 1b. Residues in units of ω_r versus ka for $m = 2,\ b/a = 2,\ \omega_p/\omega_r\ = 10$.

diocotron mode are shown. Higher modes have frequencies closer to $m\omega_r$ and even smaller residues. These fall in the shaded regions of the figures. Fast modes, those whose fields rotate faster than plasma rotation $(\omega/m > \omega_r)$ have *negative* residues, while slow modes $(\omega/m < \omega_r)$ have *positive* residues. Thus the excitation of a fast mode adds angular momentum so that a *fast wave has positive angular momentum.* Similarly, the excitation of a slow mode removes angular momentum so that a *slow wave has negative angular momentum*.

For $ka \ll 1$, the dispersion relation of the l'th radial TG eigenmode is $\omega_{lmn} \simeq m\omega_r \pm ka\,(\omega_p/\,p_{ml})$, where p_{ml} is the $l\text{-}th$ root of $J_m(x) = 0$. Similarly, $R_{kmn}\ \simeq\ C_{lm}\,\omega_r\,(ka)^3$.

A similar expression has been obtained for the susceptiblity of spheroidal plasmas, using the methods of Ref. 5. LeGendere functions replace the Bessel functions.

It has recently been shown the rotation of a Coulomb crystal can be phase-locked to a rotating wall field(6) and that this is a very useful technique. Here we calculate the torque on an ideal Coulomb crystal. The effective potential of the applied trap field is static in a frame rotating with the field and has the following form

$$\Phi_T = V_0\ [z^2 + \beta\rho^2] + V_1 \frac{P_m^m(\xi_1/d)}{P_m^m(b/d)} \frac{P_m^m(\xi_2)}{P_m^m(0)} cos(m\phi).$$

where $V_0 = m\omega_z^2/2e$, with ω_z the single particle axial bounce frequency of the trap and V_1 the amplitude of the rotating multipole potential at the crystal boundary, $\xi_1 = b$,

$\xi_2 = 0$. In equilibrium, the crystal adjusts its so as to produce a potential Φ_p which just cancels Φ_T inside the crystal. This gives a spheroid(6), of radius a and half-length b, with a slight deformation of its surface arising from the second term. The deformation can be characterized by its surface charge,

$$\sigma = n_o e a \, \delta \, \frac{h_1(\xi_2{=}0)}{h_1(\xi_2)} \, \frac{P_m^m(\xi_2)}{P_m^m(0)} cos[m(\phi + \triangle\phi)],$$

where δ is the fractional distortion in the radius a, h_1 is the element of the metric tensor associated with ξ_1. In equilibrium, $\triangle\phi = 0$, and the torque on this surface charge vanishes. However, if this charge distribution is rotated (ahead) through an angle $\triangle\phi$ from the equilibrium position *without significant change in shape*, then the torque $T = \int_V \rho E_\phi dq$ is non-zero and is

$$T = c_m Q_o V_1 \delta \, sin(m\triangle\phi) = \, - T_{max} \, sin(m\triangle\phi)$$

where Q_o is the total charge of the crystal, and $c_m = (3m/4) \int_{-1}^{+1} [P_m^m(\xi)/P_m^m(0)]^2 d\xi$ $= 8/5$ for $m = 2$. V_1 is proportional to δ so that the torque is quadratic in δ. This torque acts to speed or slow the rotation, according to the sign of $m\triangle\phi$ and keep the crystal rotating, on the average, with the applied field. However there may be small amplitude phase oscillations and there is a maximum torque T_{max} which governs the maximum rate at which the crystal rotation can be accelerated. Both also depend on the moment of inertia of the crystal, and phase oscillations are probably damped by viscous effects. Phase oscillations might be excited by an abrupt change in phase of the rotating field, and a study of the response of the system may yield useful properties of the crystal, since the time required to come into a new equilibrium shape is not known. Again, it should not be necessary for the applied field to be a "rotating wall" because the rotating crystal will respond mainly to the *rotating component* of a standing wave field whose angular velocity is close to that of the crystal.

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